MOCK THETA FUNCTIONS, RANKS, AND MAASS FORMS

KEN ONO

1. Introduction

Generating functions play a central role throughout number theory. For example in the theory of partitions, if $p(n)$ denotes the number of partitions of an integer n, then Euler observed that

$$(1.1)\quad P(q) := \sum_{n=0}^{\infty} p(n) q^{24n-1} = q^{-1} \prod_{n=1}^{\infty} \frac{1}{1-q^{24n}} = q^{-1} + q^{23} + 2q^{47} + 3q^{71} + 5q^{95} + \cdots .$$

Similarly in the theory of quadratic forms, we have the following fundamental q-series identity of Jacobi

$$(1.2)\qquad \Theta(q) := \sum_{n=-\infty}^{\infty} q^{n^2} = \prod_{n=1}^{\infty} \frac{(1-q^{2n})^5}{(1-q^n)^2(1-q^{4n})^2} = 1 + 2q + 2q^4 + 2q^9 + \cdots .$$

Consequently, it follows that the integers $r_s(n)$, the number of representations of integers n as sums of s squares, are formally given as the coefficients of the q-series

$$\sum_{n=0}^{\infty} r_s(n) q^n = \Theta(q)^s = \prod_{n=1}^{\infty} \frac{(1-q^{2n})^{5s}}{(1-q^n)^{2s}(1-q^{4n})^{2s}}.$$

As a third example, consider the q-series $A(q)$ defined by

$$(1.3)\quad A(q) := \sum_{n=1}^{\infty} a_E(n) q^n := q \prod_{n=1}^{\infty} (1-q^{4n})^2 (1-q^{8n})^2 = q - 2q^5 - 3q^9 + 6q^{13} + 2q^{17} - \cdots .$$

It is well known that $L(E,s) = \sum_{n=1}^{\infty} \frac{a_E(n)}{n^s}$ is the Hasse-Weil L-function of the elliptic curve

$$E: \quad y^2 = x^3 - x.$$

In particular, this fact implies that if p is an odd prime, then

$$a_E(p) = p - \#\{(x,y) \in (\mathbb{F}_p)^2 \ : \ y^2 \equiv x^3 - x \pmod{p}\} = -\sum_{x=0}^{p-1} \left(\frac{x^3-x}{p}\right).$$

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Therefore, the q-series $A(q)$ can be naively thought of as the "generating function" for the number of $\mathbb{F}_p$-points on the reductions of the elliptic curve E as one varies p.

These examples share the property that they all are generating functions which coincide with Fourier expansions of modular forms. Loosely speaking, a *modular form of weight k on a subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ with multiplier system ϵ* is any meromorphic function $f(z)$ on $\mathbb{H}$ with the property that

$$f\left(\frac{az+b}{cz+d}\right) = \epsilon(a,b,c,d)(cz+d)^k f(z)$$

for every $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. The modularity of these generating functions follows from the modularity of Dedekind's eta-function

$$\eta(z) := q^{1/24} \prod_{n=1}^{\infty} (1-q^n) \tag{1.4}$$

(note. $q := e^{2\pi i z}$ throughout). More precisely, modularity follows from the well known transformation laws

$$\eta(z+1) = e(1/24)\eta(z) \quad \text{ and } \quad \eta(-1/z) = (-iz)^{\frac{1}{2}}\eta(z), \tag{1.5}$$

where $e(\alpha) := e^{2\pi i \alpha}$.

In this context, the theory of modular forms has played a central role in the study of partitions, quadratic forms, and elliptic curves, as well as many other topics throughout mathematics. The rich theory of modular forms allows one to prove theorems about asymptotics, congruences, and multiplicative relations.

On the other hand, there are many related questions in which modular forms do not appear to play a role. Here we consider problems on mock theta functions and partitions, and we show that *weak Maass forms* play a prominent role. We shall consider recent works [11, 12] by Bringmann and the author on mock theta functions and Dyson's partition ranks. In particular, we shall review recent results on:

- Modularity of mock theta functions,
- Dyson's partition ranks and partition congruences,
- The Andrews-Dragonette Conjecture for the mock theta function $f(q)$.

2. Weak Maass forms

We begin by recalling the notion of a weak Maass form of half-integral weight $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$. If $z = x+iy$ with $x, y \in \mathbb{R}$, then the weight k hyperbolic Laplacian is given by

$$\Delta_k := -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right). \tag{2.1}$$

If v is odd, then define ϵ_v by

$$\epsilon_v := \begin{cases} 1 & \text{if } v \equiv 1 \pmod 4, \\ i & \text{if } v \equiv 3 \pmod 4. \end{cases} \tag{2.2}$$

A *weak Maass form of weight k on a subgroup* $\Gamma \subset \Gamma_0(4)$ is any smooth function $f : \mathbb{H} \to \mathbb{C}$ satisfying the following:

(1) For all $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and all $z \in \mathbb{H}$, we have
$$f(Az) = \left(\frac{c}{d}\right)^{2k} \epsilon_d^{-2k} (cz+d)^k \ f(z).$$
Here $\left(\frac{c}{d}\right)$ denotes the extended Legendre symbol.

(2) We have that $\Delta_k f = 0$.

(3) The function $f(z)$ has at most linear exponential growth at all the cusps of Γ.

Similarly, we have the notion of a weak Maass form with Nebentypus. To define it, suppose that N is a positive integer, and that $\psi \pmod{4N}$ is a Dirichlet character. A weight k weak Maass form on $\Gamma_1(4N)$ is a *weak Maass form on* $\Gamma_0(4N)$ *with Nebentypus character* ψ if for every $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4N)$ and all $z \in \mathbb{H}$ we have
$$f(Az) = \psi(d) \left(\frac{c}{d}\right)^{2k} \epsilon_d^{-2k} \ (cz+d)^k \ f(z).$$

Remark. The transformation laws in these definitions coincide with those in Shimura's theory of half-integral weight modular forms [31].

Weak Maass forms can be related to classical *weakly holomorphic modular forms*, those forms whose poles (if there are any) are supported at cusps. A weak Maass form which is holomorphic on $\mathbb{H}$ is already a weakly holomorphic modular form. More generally, one may relate weak Maass forms to weakly holomorphic modular forms using the anti-linear differential operator ξ_k defined by

$$\xi_k(f)(z) := 2iy^k \overline{\tfrac{\partial}{\partial \bar{z}} f(z)}. \tag{2.3}$$

Here we have that
$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

In their work on geometric theta lifts, Bruinier and Funke (see Prop. 3.2 of [13]) proved the following valuable proposition.

Proposition 2.1. *If $f(z)$ is a weak Maass form of weight k for the group $\Gamma_0(4N)$ with Nebentypus χ, then $\xi_k(f)$ is a weakly holomorphic modular form of weight $2-k$ on $\Gamma_0(4N)$ with Nebentypus χ. Furthermore, ξ_k has the property that its kernel consists of those weight k weak Maass forms which are weakly holomorphic modular forms.*

Remark. Proposition 2.1 holds for weak Maass forms on any subgroup $\Gamma \subset \Gamma_0(4)$, not just those with Nebentypus.

The purpose of this expository paper is to describe a number of instances where the theory of weak Maass forms gives new results on mock theta functions and partition ranks. In particular, it will turn out that well known generating functions appear as "pieces" of weak Maass forms. Weak Maass forms have Fourier expansions of the form

$$f(z) = \sum_{n=n_0}^{\infty} \gamma(f,n;y)q^{-n} + \sum_{n=n_1}^{\infty} a(f,n)q^n.$$

As one sees, the Fourier coefficients $\gamma(f,n;y)$ are functions in y, the imaginary part of z, while the coefficients $a(f,n)$ are ordinary complex numbers. Therefore, we shall refer to $\sum_{n=n_0}^{\infty} \gamma(f,n;y)q^{-n}$ as the "non-holomorphic part" of $f(z)$, and we shall refer to $\sum_{n=n_1}^{\infty} a(f,n)q^n$ as its "holomorphic part." The number theoretic generating functions we consider are holomorphic parts of weak Maass forms.

3. Mock theta functions and partition ranks

"*The mock theta-functions give us tantalizing hints of a grand synthesis still to be discovered. Somehow it should be possible to build them into a coherent group-theoretical structure, analogous to the structure of modular forms which Hecke built around the old theta-functions of Jacobi. This remains a challenge for the future. My dream is that I will live to see the day when our young physicists, struggling to bring the predictions of superstring theory into correspondence with the facts of nature, will be led to enlarge their analytic machinery to include not only theta-functions but mock theta-functions. ... But before this can happen, the purely mathematical exploration of the mock-modular forms and their mock-symmetries must be carried a great deal further.*"

Freeman Dyson, 1987
Ramanujan Centenary Conference

Dyson's quote (see page 20 of [18]) refers to 22 peculiar q-series, such as

$$f(q) := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2 \cdots (1+q^n)^2}, \tag{3.1}$$

which were defined by Ramanujan and Watson decades ago. In his last letter to Hardy dated January 1920 (see pages 127-131 of [28]), Ramanujan lists 17 such functions, and he gives 2 more in his "Lost Notebook" [28]. In his paper "The final problem: an account of the mock theta functions" [33], Watson defines 3 further functions. Although much remains unknown about these enigmatic series, Ramanujan's mock theta functions have been the subject of an astonishing number of important works (for example, see [3, 4, 6, 7, 14, 15, 16, 20, 21, 22, 23, 28, 29, 33, 34, 35, 36] to name a few).

In his 2002 Ph.D. thesis [36], written under the direction of Zagier, Zwegers made an important step in the direction of Dyson's "challenge for the future" by relating many of Ramanujan's mock theta functions to real analytic vector valued modular forms. More recently, Bringmann and the author have shown [12] that Dyson's own rank generating function can already be used to construct the desired "coherent group-theoretical structure, analogous to the structure of modular forms which Hecke built around old theta functions of Jacobi". More precisely, we relate specializations of his partition rank generating function to weak Maass forms, and we show that the non-holomorphic parts of these forms are period integrals of theta functions, thereby realizing Dyson's speculation that such a picture should involve theta functions.

In an effort to provide a combinatorial explanation of Ramanujan's congruences for $p(n)$, Dyson introduced [17] the so-called "rank" of a partition. The rank of a partition is defined to be its largest part minus the number of its parts. If $N(m,n)$ denotes the number of partitions of n with rank m, then it is well known that

$$R(w;q) := 1 + \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} N(m,n) w^m q^n = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(wq;q)_n (w^{-1}q;q)_n}, \tag{3.2}$$

where

$$(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}),$$
$$(a;q)_\infty := \prod_{m=0}^{\infty} (1-aq^m).$$

Letting $w=-1$, we obtain the series

$$R(-1;q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2}.$$

This is the mock theta function $f(q)$ given in (3.1). This observation connects the additive number theory of partitions to mock theta functions. In the next section, we link the specializations of these generating functions, at roots of unity $w \neq 1$, to weight $1/2$ weak Maass forms.

3.1. **Dyson's generating functions and Maass forms.** Here we complete Dyson's generating functions to obtain weak Maass forms. Suppose that $0 < a < c$ are integers, and let $\zeta_c := e^{2\pi i/c}$. If $f_c := \frac{2c}{\gcd(c,6)}$, then define the theta function $\Theta\left(\frac{a}{c};\tau\right)$ by

$$\Theta\left(\frac{a}{c};\tau\right) := \sum_{m \pmod{f_c}} (-1)^m \sin\left(\frac{a\pi(6m+1)}{c}\right) \cdot \theta\left(6m+1, 6f_c; \frac{\tau}{24}\right), \tag{3.3}$$

where $\tau \in \mathbb{H}$ and

$$\theta(\alpha,\beta;\tau) := \sum_{n \equiv \alpha \pmod{\beta}} n e^{2\pi i \tau n^2}. \tag{3.4}$$

Throughout, let $\ell_c := \mathrm{lcm}(2c^2, 24)$, and let $\widetilde{\ell_c} := \ell_c/24$. It is well known [31] that $\Theta\left(\frac{a}{c}; \ell_c\tau\right)$ is a weight 3/2 cusp form, and we use it to define the function $S_1\left(\frac{a}{c}; z\right)$

$$S_1\left(\frac{a}{c}; z\right) := \frac{-i\sin\left(\frac{\pi a}{c}\right)\ell_c^{\frac{1}{2}}}{\sqrt{3}} \int_{-\bar{z}}^{i\infty} \frac{\Theta\left(\frac{a}{c}; \ell_c\tau\right)}{\sqrt{-i(\tau+z)}}\, d\tau. \tag{3.5}$$

Using this notation, define $D\left(\frac{a}{c}; z\right)$ by

$$D\left(\frac{a}{c}; z\right) := -S_1\left(\frac{a}{c}; z\right) + q^{-\frac{\ell_c}{24}} R(\zeta_c^a; q^{\ell_c}). \tag{3.6}$$

Theorem 3.1. (Bringmann-Ono, Theorems 1.1 and 1.2 of [12])
If $0 < a < c$, then $D\left(\frac{a}{c}; z\right)$ is a weight 1/2 weak Maass form on Γ_c, where

$$\Gamma_c := \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \ell_c^2 & 1 \end{pmatrix} \right\rangle. \tag{3.7}$$

Moreover, if c is odd, then $D\left(\frac{a}{c}; z\right)$ is a weak Maass form of weight 1/2 on $\Gamma_1(144 f_c^2 \widetilde{\ell_c})$.

Sketch of the proof. The conclusion of the theorem follows from a general result (see Theorem 3.4 of [12]) about vector valued weight 1/2 weak Maass forms for the modular group $\mathrm{SL}_2(\mathbb{Z})$, a result which is of independent interest. For brevity, here we only sketch the proof of the first claim.

To prove this claim, we require modular transformation laws for the series $R(\zeta_c^a; q^{\ell_c})$. Some of these laws have been obtained recently by Gordon and McIntosh [21]. To state their results, we first define the series

(3.8)

$$M\left(\frac{a}{c}; z\right) = M\left(\frac{a}{c}; q\right) := \frac{1}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n+\frac{a}{c}}}{1-q^{n+\frac{a}{c}}} \cdot q^{\frac{3}{2}n(n+1)},$$

$$M_1\left(\frac{a}{c}; z\right) = M_1\left(\frac{a}{c}; q\right) := \frac{1}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1} q^{n+\frac{a}{c}}}{1+q^{n+\frac{a}{c}}} \cdot q^{\frac{3}{2}n(n+1)},$$

$$N\left(\frac{a}{c}; z\right) = N\left(\frac{a}{c}; q\right) := \frac{1}{(q;q)_\infty} \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n (1+q^n)\left(2-2\cos\left(\frac{2\pi a}{c}\right)\right)}{1-2q^n\cos\left(\frac{2\pi a}{c}\right)+q^{2n}} \cdot q^{\frac{n(3n+1)}{2}}\right),$$

$$N_1\left(\frac{a}{c}; z\right) = N_1\left(\frac{a}{c}; q\right) := \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n (1-q^{2n+1})}{1-2q^{n+\frac{1}{2}}\cos\left(\frac{2\pi a}{c}\right)+q^{2n+1}} \cdot q^{\frac{3n(n+1)}{2}}.$$

Gordon and McIntosh show the following q-series identities

$$M\left(\frac{a}{c}; q\right) = \sum_{n=1}^{\infty} \frac{q^{n(n-1)}}{(q^{\frac{a}{c}};q)_n \cdot (q^{1-\frac{a}{c}};q)_n}, \tag{3.9}$$

$$N\left(\frac{a}{c};q\right)=1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{\prod_{j=1}^{n}\left(1-2\cos\left(\frac{2\pi a}{c}\right)q^j+q^{2j}\right)}. \tag{3.10}$$

Obviously, (3.8) and (3.10) imply the important fact that

$$R(\zeta_c^a;q)=N\left(\frac{a}{c};q\right). \tag{3.11}$$

Their transformation laws involve the following Mordell integrals

$$\begin{aligned} J\left(\frac{a}{c};\alpha\right)&:=\int_0^\infty e^{-\frac{3}{2}\alpha x^2}\cdot\frac{\cosh\left(\left(\frac{3a}{c}-2\right)\alpha x\right)+\cosh\left(\left(\frac{3a}{c}-1\right)\alpha x\right)}{\cosh(3\alpha x/2)}\,dx,\\ J_1\left(\frac{a}{c};\alpha\right)&:=\int_0^\infty e^{-\frac{3}{2}\alpha x^2}\cdot\frac{\sinh\left(\left(\frac{3a}{c}-2\right)\alpha x\right)-\sinh\left(\left(\frac{3a}{c}-1\right)\alpha x\right)}{\sinh(3\alpha x/2)}\,dx.\end{aligned} \tag{3.12}$$

Suppose that α and β have the property that $\alpha\beta=\pi^2$. Gordon and McIntosh then prove (see page 199 of [21]) that if $q:=e^{-\alpha}$ and $q_1:=e^{-\beta}$, then

$$\begin{aligned} q^{\frac{3a}{2c}\left(1-\frac{a}{c}\right)-\frac{1}{24}}\cdot M\left(\frac{a}{c};q\right)&=\sqrt{\frac{\pi}{2\alpha}}\csc\left(\frac{a\pi}{c}\right)q_1^{-\frac{1}{6}}\cdot N\left(\frac{a}{c};q_1^4\right)-\sqrt{\frac{3\alpha}{2\pi}}\cdot J\left(\frac{a}{c};\alpha\right),\\ q^{\frac{3a}{2c}\left(1-\frac{a}{c}\right)-\frac{1}{24}}\cdot M_1\left(\frac{a}{c};q\right)&=-\sqrt{\frac{2\pi}{\alpha}}q_1^{\frac{4}{3}}\cdot N_1\left(\frac{a}{c};q_1^2\right)-\sqrt{\frac{3\alpha}{2\pi}}\cdot J_1\left(\frac{a}{c};\alpha\right).\end{aligned} \tag{3.13}$$

Using the functions

$$\mathcal{N}\left(\frac{a}{c};q\right)=\mathcal{N}\left(\frac{a}{c};z\right):=\csc\left(\frac{a\pi}{c}\right)\cdot q^{-\frac{1}{24}}\cdot N\left(\frac{a}{c};q\right), \tag{3.14}$$

$$\mathcal{M}\left(\frac{a}{c};q\right)=\mathcal{M}\left(\frac{a}{c};z\right):=2q^{\frac{3a}{2c}\cdot\left(1-\frac{a}{c}\right)-\frac{1}{24}}\cdot M\left(\frac{a}{c};q\right), \tag{3.15}$$

define the vector valued function $F\left(\frac{a}{c};z\right)$ by

$$\begin{aligned} F\left(\frac{a}{c};z\right)&:=\left(F_1\left(\frac{a}{c};z\right),F_2\left(\frac{a}{c};z\right)\right)^T\\ &=\left(\sin\left(\frac{\pi a}{c}\right)\mathcal{N}\left(\frac{a}{c};\ell_c z\right),\sin\left(\frac{\pi a}{c}\right)\mathcal{M}\left(\frac{a}{c};\ell_c z\right)\right)^T.\end{aligned} \tag{3.16}$$

Similarly, define the vector valued (non-holomorphic) function $G\left(\frac{a}{c};z\right)$ by

(3.17)

$$\begin{aligned} G\left(\frac{a}{c};z\right)&=\left(G_1\left(\frac{a}{c};z\right),G_2\left(\frac{a}{c};z\right)\right)^T\\ &:=\left(2\sqrt{3}\sin\left(\frac{\pi a}{c}\right)\sqrt{-i\ell_c z}\cdot J\left(\frac{a}{c};-2\pi i\ell_c z\right),\frac{2\sqrt{3}\sin\left(\frac{\pi a}{c}\right)}{i\ell_c z}\cdot J\left(\frac{a}{c};\frac{2\pi i}{\ell_c z}\right)\right)^T.\end{aligned}$$

The transformations in (3.13) easily imply the following transformations under the generators of Γ_c.

Lemma 3.2. *Assume the notation and hypotheses above. For $z \in \mathbb{H}$, we have*

$$\begin{aligned} F\left(\frac{a}{c}; z+1\right) &= F\left(\frac{a}{c}; z\right), \\ \frac{1}{\sqrt{-i\ell_c z}} \cdot F\left(\frac{a}{c}; -\frac{1}{{\ell_c}^2 z}\right) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot F\left(\frac{a}{c}; z\right) + G\left(\frac{a}{c}; z\right). \end{aligned}$$

The Mordell vector $G\left(\frac{a}{c}; z\right)$ arises as integrals of the theta function $\Theta\left(\frac{a}{c}; \tau\right)$.

Lemma 3.3. *Assume the notation and hypotheses above. For $z \in \mathbb{H}$, we have*

$$G\left(\frac{a}{c}; z\right) = \frac{i\ell_c^{\frac{1}{2}} \sin\left(\frac{\pi a}{c}\right)}{\sqrt{3}} \int_0^{i\infty} \frac{\left((-i\ell_c\tau)^{-\frac{3}{2}}\Theta\left(\frac{a}{c}; -\frac{1}{\ell_c\tau}\right), \Theta\left(\frac{a}{c}; \ell_c\tau\right), \right)^T}{\sqrt{-i(\tau+z)}}\, d\tau.$$

Proof of the lemma. For brevity, we only prove the asserted formula for the first component of $G\left(\frac{a}{c}; z\right)$. The proof of the second component follows in the same way.

By analytic continuation and a change of variables (note. we may assume that $z = it$ with $t > 0$), we find that

$$J\left(\frac{a}{c}; \frac{2\pi}{\ell_c t}\right) = \ell_c t \cdot \int_0^{\infty} e^{-3\ell_c \pi t x^2} \cdot \frac{\cosh\left(\left(\frac{3a}{c} - 2\right) 2\pi x\right) + \cosh\left(\left(\frac{3a}{c} - 1\right) 2\pi x\right)}{\cosh(3\pi x)}\, dx.$$

Using the Mittag-Leffler theory of partial fraction decompositions, one finds that

$$\begin{aligned} &\frac{\cosh\left(\left(\frac{3a}{c} - 2\right) 2\pi x\right) + \cosh\left(\left(\frac{3a}{c} - 1\right) 2\pi x\right)}{\cosh(3\pi x)} \\ &\qquad = \frac{-i}{\sqrt{3}\pi} \sum_{n\in\mathbb{Z}} \frac{(-1)^n \sin\left(\frac{\pi a(6n+1)}{c}\right)}{x - i\left(n + \frac{1}{6}\right)} - \frac{i}{\sqrt{3}\pi} \sum_{n\in\mathbb{Z}} \frac{(-1)^n \sin\left(\frac{\pi a(6n+1)}{c}\right)}{-x - i\left(n + \frac{1}{6}\right)}. \end{aligned}$$

By introducing the extra term $\frac{1}{i\left(n+\frac{1}{6}\right)}$, we just have to consider

$$\int_{-\infty}^{\infty} e^{-3\pi\ell_c t x^2} \sum_{n\in\mathbb{Z}} (-1)^n \sin\left(\frac{\pi a(6n+1)}{c}\right) \left(\frac{1}{x - i\left(n+\frac{1}{6}\right)} + \frac{1}{i\left(n+\frac{1}{6}\right)}\right) dx.$$

Since this expression is absolutely convergent, we may interchange summation and integration to obtain

$$J\left(\frac{a}{c}; \frac{2\pi}{\ell_c t}\right) = \frac{-\ell_c i t}{\sqrt{3}\pi} \sum_{n\in\mathbb{Z}} (-1)^n \sin\left(\frac{\pi a(6n+1)}{c}\right) \int_{-\infty}^{\infty} \frac{e^{-3\pi\ell_c t x^2}}{x - i\left(n+\frac{1}{6}\right)}\, dx.$$

For all $s \in \mathbb{R} \setminus \{0\}$, we have the identity

$$\int_{-\infty}^{\infty} \frac{e^{-\pi t x^2}}{x - is}\, dx = \pi i s \int_0^{\infty} \frac{e^{-\pi u s^2}}{\sqrt{u+t}}\, du$$

(this follows since both sides are solutions of $\left(-\frac{\partial}{\partial t}+\pi s^2\right)f(t)=\frac{\pi i s}{\sqrt{t}}$ and have the same limit 0 as $t\mapsto\infty$ and hence are equal). Hence we may conclude that

$$J\left(\frac{a}{c};\frac{2\pi}{\ell_c t}\right)=\frac{\ell_c t}{6\sqrt{3}}\sum_{n\in\mathbb{Z}}(-1)^n(6n+1)\sin\left(\frac{\pi a(6n+1)}{c}\right)\int_0^\infty\frac{e^{-\pi(n+1/6)^2u}}{\sqrt{u+3\ell_c t}}\,du.$$

Substituting $u=-3\ell_c i\tau$, and interchanging summation and integration gives

$$J\left(\frac{a}{c};\frac{2\pi}{\ell_c t}\right)=\frac{-it{\ell_c}^{\frac{3}{2}}}{6}\int_0^{i\infty}\frac{\sum_{n\in\mathbb{Z}}(-1)^n(6n+1)\sin\left(\frac{\pi a(6n+1)}{c}\right)e^{3\pi i\ell_c\tau\left(n+\frac{1}{6}\right)^2}}{\sqrt{-i(\tau+it)}}\,d\tau.$$

The claim follows since the sum over n coincides with definition (3.3). $\square$

We must determine the necessary modular transformation properties of the vector

$$\begin{aligned}S\left(\frac{a}{c};z\right)&=\left(S_1\left(\frac{a}{c};z\right),S_2\left(\frac{a}{c};z\right)\right)\\&:=\frac{-i\sin\left(\frac{\pi a}{c}\right){\ell_c}^{\frac{1}{2}}}{\sqrt{3}}\int_{-\bar{z}}^{i\infty}\frac{\left(\Theta\left(\frac{a}{c};\ell_c\tau\right),(-i\ell_c\tau)^{-\frac{3}{2}}\Theta\left(\frac{a}{c};-\frac{1}{\ell_c\tau}\right)\right)^T}{\sqrt{-i(\tau+z)}}\,d\tau.\end{aligned}\tag{3.18}$$

Since $\Theta\left(\frac{a}{c};\ell_c z\right)$ is a cusp form, the integral above is absolutely convergent. The next lemma shows that $S\left(\frac{a}{c};z\right)$ satisfies the same transformations as $F\left(\frac{a}{c};z\right)$.

Lemma 3.4. *Assume the notation and hypotheses above. For $z\in\mathbb{H}$, we have*

$$\begin{aligned}S\left(\frac{a}{c};z+1\right)&=S\left(\frac{a}{c};z\right),\\\frac{1}{\sqrt{-i\ell_c z}}\cdot S\left(\frac{a}{c};-\frac{1}{{\ell_c}^2z}\right)&=\begin{pmatrix}0&1\\1&0\end{pmatrix}\cdot S\left(\frac{a}{c};z\right)+G\left(\frac{a}{c};z\right).\end{aligned}$$

Proof of the lemma. Using the Fourier expansion of $\Theta\left(\frac{a}{c};z\right)$, one easily sees that

$$S_1\left(\frac{a}{c};z+1\right)=S_1\left(\frac{a}{c};z\right).$$

Using classical facts about theta functions [31], we also have that

$$S_2\left(\frac{a}{c};z+1\right)=S_2\left(\frac{a}{c};z\right).$$

Hence, it suffices to prove the second transformation law. We directly compute

$$\begin{aligned}&\frac{1}{\sqrt{-i\ell_c z}}\cdot S\left(\frac{a}{c};-\frac{1}{{\ell_c}^2z}\right)\\&\quad=\frac{i\sin\left(\frac{\pi a}{c}\right){\ell_c}^{\frac{1}{2}}}{\sqrt{3}\sqrt{-i\ell_c z}}\int_{\frac{1}{{\ell_c}^2\bar{z}}}^{i\infty}\frac{\left(\Theta\left(\frac{a}{c};\ell_c\tau\right),(-i\ell_c\tau)^{-\frac{3}{2}}\Theta\left(\frac{a}{c};-\frac{1}{\ell_c\tau}\right)\right)^T}{\sqrt{-i\left(\tau-\frac{1}{{\ell_c}^2z}\right)}}\,d\tau,\end{aligned}$$

and after making the change of variable $\tau \mapsto -\frac{1}{{\ell_c}^2\tau}$, we obtain

$$\begin{aligned}\frac{1}{\sqrt{-i\ell_c z}} &\cdot S\left(\frac{a}{c}; -\frac{1}{{\ell_c}^2 z}\right)\\ &= \frac{i\sin\left(\frac{\pi a}{c}\right){\ell_c}^{\frac{1}{2}}}{\sqrt{3}} \int_0^{-\bar{z}} \frac{\left((-i\ell_c\tau)^{-\frac{3}{2}}\Theta\left(\frac{a}{c}; -\frac{1}{\ell_c\tau}\right), \Theta\left(\frac{a}{c}, \ell_c\tau\right)\right)^T}{\sqrt{-i(\tau+z)}}\, d\tau.\end{aligned}$$

Consequently, we obtain the desired conclusion

$$\begin{aligned}\frac{1}{\sqrt{-i\ell_c z}} &\cdot S\left(\frac{a}{c}; -\frac{1}{{\ell_c}^2 z}\right) - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot S\left(\frac{a}{c}; z\right)\\ &= \frac{i\sin\left(\frac{\pi a}{c}\right){\ell_c}^{\frac{1}{2}}}{\sqrt{3}} \int_0^{i\infty} \frac{\left((-i\ell_c\tau)^{-\frac{3}{2}}\Theta\left(\frac{a}{c}; -\frac{1}{\ell_c\tau}\right), \Theta\left(\frac{a}{c}; \ell_c\tau\right)\right)^T}{\sqrt{-i(\tau+z)}}\, d\tau = G\left(\frac{a}{c}; z\right).\end{aligned}$$

□

Using (3.6), (3.8), (3.11), (3.14), and (3.16), we find that we have already determined the transformation laws satisfied by $D\left(\frac{a}{c}; z\right)$ since we have

$$\begin{pmatrix} 1 & 0 \\ {\ell_c}^2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -{\ell_c}^2 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -\frac{1}{{\ell_c}^2} \\ 1 & 0 \end{pmatrix}.$$

The key point is that the first and third matrices on the right provide the same Möbius transformation on $\mathbb{H}$. Therefore the transformation laws for $D\left(\frac{a}{c}; z\right)$ follow from Lemma 3.2 and Lemma 3.4.

Now we show that $D\left(\frac{a}{c}; z\right)$ is annihilated by

$$\Delta_{\frac{1}{2}} = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{iy}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) = -4y^{\frac{3}{2}}\frac{\partial}{\partial z}\sqrt{y}\frac{\partial}{\partial \bar{z}}.$$

Since $q^{-\frac{\ell_c}{24}}R(\zeta_b^a; q^{\ell_c})$ is a holomorphic function in z, we get

$$\frac{\partial}{\partial \bar{z}}\left(D\left(\frac{a}{c}; z\right)\right) = -\frac{\partial}{\partial \bar{z}}\left(S_1\left(\frac{a}{c}; z\right)\right) = \frac{\sin\left(\frac{\pi a}{c}\right)\ell_c^{\frac{1}{2}}}{\sqrt{6y}} \cdot \Theta\left(\frac{a}{c}; -\ell_c\bar{z}\right).$$

Hence, we find that $\sqrt{y}\frac{\partial}{\partial \bar{z}}\left(D\left(\frac{a}{c}; z\right)\right)$ is anti-holomorphic, and so

$$\frac{\partial}{\partial z}\sqrt{y}\frac{\partial}{\partial \bar{z}}\left(D\left(\frac{a}{c}; z\right)\right) = 0.$$

To complete the proof, it suffices to show that $D\left(\frac{a}{c}; z\right)$ has at most linear exponential growth at cusps. This follows from the convergence of the period integral $S_1\left(\frac{a}{c}; z\right)$ and the transformation laws satisfied by $D\left(\frac{a}{c}; z\right)$. □

Remark. Zwegers observed [35] that one may interprete Watson's transformation laws for the third order mock theta functions in terms of a vector valued real analytic modular form on $\mathrm{SL}_2(\mathbb{Z})$. Bringmann and the author constructed (see Theorem 3.4 of [12]) an infinite class of $\mathrm{SL}_2(\mathbb{Z})$ vector valued weight 1/2 weak Maass forms. This result implies Theorem 3.1, and it is a generalization of Zwegers' observation.

3.2. **Ramanujan's partition congruences.** Theorem 3.1 sheds new light on the role that Dyson's rank plays in the theory of partition congruences. If r and t are integers, then let $N(r,t;n)$ be the number of partitions of n whose rank is $r \pmod t$. Using a standard argument involving the orthogonality relations on sums of roots of unity, it is straightforward to deduce that if $0 \le r < t$ are integers, then

$$\sum_{n=0}^{\infty} N(r,t;n)q^n = \frac{1}{t}\sum_{n=0}^{\infty} p(n)q^n + \frac{1}{t}\sum_{j=1}^{t-1} \zeta_t^{-rj} \cdot R(\zeta_t^j;q). \tag{3.19}$$

By Theorem 3.1, it then follows that

$$\sum_{n=0}^{\infty} \left(N(r,t;n) - \frac{p(n)}{t} \right) q^{\ell_t n - \frac{\ell_t}{24}}$$

is the holomorphic part of a weak Maass form of weight 1/2 on Γ_t, one which is given as an appropriate weighted sum of the weak Maass forms $D\left(\frac{a}{t};z\right)$. If t is odd, then it is on $\Gamma_1(144 f_t^2 \widetilde{\ell_t})$. This result allows us to relate many "sieved" generating functions to weakly holomorphic modular forms.

Theorem 3.5. (Bringmann-Ono, Theorem 1.4 of [12])
If $0 \le r < t$ are integers, where t is odd, and $\mathcal{P} \nmid 6t$ is prime, then

$$\sum_{\substack{n \ge 1 \\ \left(\frac{24\ell_t n - \ell_t}{\mathcal{P}}\right) = -\left(\frac{-24\widetilde{\ell_t}}{\mathcal{P}}\right)}} \left(N(r,t;n) - \frac{p(n)}{t} \right) q^{\ell_t n - \frac{\ell_t}{24}}$$

is a weight 1/2 weakly holomorphic modular form on $\Gamma_1(144 f_t^2 \widetilde{\ell_t} \mathcal{P}^4)$.

Sketch of the proof. The proof requires the Fourier expansions of the forms $D\left(\frac{a}{c};z\right)$. To give these expansions, we require the incomplete Gamma-function

$$\Gamma(a;x) := \int_x^{\infty} e^{-t} t^{a-1}\, dt. \tag{3.20}$$

For integers $0 < a < c$, we have

(3.21)
$$D\left(\frac{a}{c};z\right) = q^{-\frac{\ell_c}{24}} + \sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} N(m,n)\zeta_c^{am} q^{\ell_c n - \frac{\ell_c}{24}}$$
$$+ \frac{i\sin\left(\frac{\pi a}{c}\right)\ell_c^{\frac{1}{2}}}{\sqrt{3}} \sum_{m \pmod{f_c}} (-1)^m \sin\left(\frac{a\pi(6m+1)}{c}\right) \sum_{n\equiv 6m+1 \pmod{6f_c}} \gamma(c,y;n) q^{-\widetilde{\ell}_c n^2},$$

where
$$\gamma(c,y;n) := \frac{i}{\sqrt{2\pi\widetilde{\ell}_c}} \cdot \Gamma\left(\frac{1}{2}; 4\pi\widetilde{\ell}_c n^2 y\right).$$

This expansion follows easily from
$$-S_1\left(\frac{a}{c};z\right) = \frac{i\sin\left(\frac{\pi a}{c}\right)\ell_c^{\frac{1}{2}}}{\sqrt{3}} \sum_{m \pmod{f_c}} (-1)^m \sin\left(\frac{a\pi(6m+1)}{c}\right)$$
$$\times \sum_{n\equiv 6m+1 \pmod{6f_c}} \int_{-\overline{z}}^{i\infty} \frac{n e^{2\pi i n^2 \widetilde{\ell}_c \tau}}{\sqrt{-i(\tau+z)}}\, d\tau,$$

and the integral identity
$$\int_{-\overline{z}}^{i\infty} \frac{n e^{2\pi i n^2 \widetilde{\ell}_c \tau}}{\sqrt{-i(\tau+z)}}\, d\tau = \gamma(c,y;n)\cdot q^{-\widetilde{\ell}_c n^2}.$$

The key point of the proof is that the non-holomorphic parts of these weak Maass forms have the property that their coefficients are supported on a fixed square class, one which is easily annihilated by taking linear combinations of quadratic twists. In particular, suppose that $\mathcal{P} \nmid 6c$ is prime. For this prime $\mathcal{P}$, let
$$g := \sum_{v=1}^{\mathcal{P}-1} \left(\frac{v}{\mathcal{P}}\right) e^{\frac{2\pi i v}{\mathcal{P}}}$$
be the usual Gauss sum with respect to $\mathcal{P}$. Define the function $D\left(\frac{a}{c};z\right)_{\mathcal{P}}$ by

(3.22)
$$D\left(\frac{a}{c};z\right)_{\mathcal{P}} := \frac{g}{\mathcal{P}} \sum_{v=1}^{\mathcal{P}-1} \left(\frac{v}{\mathcal{P}}\right) D\left(\frac{a}{c};z\right) \ |_{\frac{1}{2}} \begin{pmatrix} 1 & -\frac{v}{\mathcal{P}} \\ 0 & 1 \end{pmatrix},$$

where $|_{\frac{1}{2}}$ is the usual "slash operator" (for example, see page 51 of [26]). By construction, $D\left(\frac{a}{c};z\right)_{\mathcal{P}}$ is the $\mathcal{P}$ quadratic twist of $D\left(\frac{a}{c};z\right)$. In other words, the nth coefficient in the q-expansion of $D\left(\frac{a}{c};z\right)_{\mathcal{P}}$ is $\left(\frac{n}{\mathcal{P}}\right)$ times the nth coefficient of $D\left(\frac{a}{c};z\right)$. For the non-holomorphic part, this follows from the fact that the factors $\gamma(c,y;n)$ appearing in (3.21) are fixed by the transformations in (3.22).

Generalizing classical facts about twists of modular forms, $D\left(\frac{a}{c};z\right)_{\mathcal{P}}$ is a weak Maass form of weight 1/2 on $\Gamma_1(144f_c^2\widetilde{\ell}_c\mathcal{P}^2)$. By (3.21), it follows that

$$D\left(\frac{a}{c};z\right)-\left(\frac{-\widetilde{\ell}_c}{\mathcal{P}}\right)D\left(\frac{a}{c};z\right)_{\mathcal{P}} \tag{3.23}$$

is a weak Maass form of weight 1/2 on $\Gamma_1(144f_c^2\widetilde{\ell}_c\mathcal{P}^2)$ with the property that its non-holomorphic part is supported on summands of the form $*q^{-\widetilde{\ell}_c\mathcal{P}^2n^2}$. These terms are annihilated by taking the $\mathcal{P}$-quadratic twist of this Maass form. Consequently, we obtain a weakly holomorphic modular form of weight 1/2 on $\Gamma_1(144f_c^2\widetilde{\ell}_c\mathcal{P}^4)$. Thanks to (3.19), when $t=c$, the conclusion of Theorem 3.5 follows. $\square$

Theorem 3.5 allows us to employ the rich theory of weakly holomorphic modular forms in the study of partition ranks. Here we describe results which were originally inspired by the celebrated Ramanujan congruences:

$$\begin{aligned} p(5n+4)&\equiv 0 \pmod{5},\\ p(7n+5)&\equiv 0 \pmod{7},\\ p(11n+6)&\equiv 0 \pmod{11}. \end{aligned}$$

In his famous seminal paper [17], Dyson conjectured that ranks could be used to provide a combinatorial "explanation" for the first two congruences[1]. Precisely, he conjectured[2] that for every integer n and every r we have

$$N(r,5;5n+4)=\frac{p(5n+4)}{5}, \tag{3.24}$$

$$N(r,7;7n+5)=\frac{p(7n+5)}{7}. \tag{3.25}$$

In an important paper, Atkin and Swinnerton-Dyer [9] confirmed Dyson's conjecture in 1954. It is not difficult to use Theorem 3.1 to give alternative proofs of these rank identities, as well as others of similar type.

[1]He further postulated the existence of another statistic, the so-called "crank" that could be used to provide an explanation for all three Ramanujan congruences. In 1988, Andrews and Garvan [8] found the crank, and they confirmed Dyson's speculation that it explains the three Ramanujan congruences. Recent work of Mahlburg [24] establishes that the Andrews-Dyson-Garvan crank plays an even more central role in the theory partition congruences. His work establishes congruences modulo arbitrary powers of all primes ≥ 5. Other work by Garvan, Kim and Stanton [19] gives a different "crank" for several other Ramanujan congruences.

[2]A short calculation reveals that this phenomenon cannot hold modulo 11.

Although identities such as (3.24) and (3.25) are rare, it is still natural to use Theorem 3.5 to investigate the relation between ranks and generic partition congruences such as

$$\begin{aligned}
&p(48037937n + 1122838) \equiv 0 \pmod{17},\\
&p(1977147619n + 815655) \equiv 0 \pmod{19},\\
&p(14375n + 3474) \equiv 0 \pmod{23},\\
&p(348104768909n + 43819835) \equiv 0 \pmod{29},\\
&p(4063467631n + 30064597) \equiv 0 \pmod{31},
\end{aligned}$$

which are now known to exist (for example, see [10, 25, 2, 1]). We shall employ a method first used by the author in [25] in his work on $p(n)$, and we show that Dyson's rank partition functions themselves uniformly satisfy many congruences of Ramanujan type.

Theorem 3.6. (Bringmann-Ono, Theorem 1.5 of [12])
Let t be a positive odd integer, and let $Q \nmid 6t$ be prime. If j is a positive integer, then there are infinitely many non-nested arithmetic progressions $An+B$ such that for every $0 \leq r < t$ we have

$$N(r,t;An+B) \equiv 0 \pmod{Q^j}.$$

Two remarks.
1) Theorem 3.6 provides a combinatorial decomposition of the partition function congruence

$$p(An+B) \equiv 0 \pmod{Q^j}.$$

2) By "non-nested", we mean that there are infinitely many arithmetic progressions $An+B$, with $0 \leq B < A$, with the property that there are no progressions which contain another progression.

Unlike Dyson's original conjectures for the congruences with modulus 5 and 7, and the work of Mahlburg [24], Theorem 3.6 says nothing about those primes $Q \geq 5$ which may divide t. It is nearly certain that this is a consequence of a non-optimal proof. There is good reason to suspect the truth of the following conjecture.

Conjecture. *Theorem 3.6 holds for those primes $Q \geq 5$ which divide t.*

Sketch of the proof of Theorem 3.6. The proof depends on Theorem 3.5, the observation that certain "sieved" partition rank generating functions are weakly holomorphic modular forms. In short, this result reduces the proof of Theorem 3.6 to the fact that any finite number of half-integral weight cusp forms with integer coefficients are annihilated, modulo a fixed prime power, by a positive proportion of half-integral weight Hecke operators.

To be precise, suppose that $f_1(z), f_2(z), \ldots, f_s(z)$ are half-integral weight cusp forms where

$$f_i(z) \in S_{\lambda_i+\frac{1}{2}}(\Gamma_1(4N_i)) \cap \mathcal{O}_K[[q]],$$

and where $\mathcal{O}_K$ is the ring of integers of a fixed number field K. If Q is prime and $j \geq 1$ is an integer, then the set of primes L for which

$$f_i(z) \mid T_{\lambda_i}(L^2) \equiv 0 \pmod{Q^j}, \tag{3.26}$$

for each $1 \leq i \leq s$, has positive Frobenius density. Here $T_{\lambda_i}(L^2)$ denotes the usual L^2 index Hecke operator of weight $\lambda_i + \frac{1}{2}$.

Suppose that $\mathcal{P} \nmid 6tQ$ is prime. By Theorem 3.5, for every $0 \leq r < t$

$$F(r,t,\mathcal{P};z) = \sum_{n=1}^{\infty} a(r,t,\mathcal{P};n)q^n := \sum_{\left(\frac{24\ell_t n - \ell_t}{\mathcal{P}}\right) = -\left(\frac{-24\widetilde{\ell}_t}{\mathcal{P}}\right)} \left(N(r,t;n) - \frac{p(n)}{t}\right) q^{\ell_t n - \frac{\ell_t}{24}} \tag{3.27}$$

is a weakly holomorphic modular form of weight $1/2$ on $\Gamma_1(144 f_t^2 \widetilde{\ell}_t \mathcal{P}^4)$. Furthermore, by the work of Ahlgren and the author [2], it follows that

$$P(t,\mathcal{P};z) = \sum_{n=1}^{\infty} p(t,\mathcal{P};n)q^n := \sum_{\left(\frac{24\ell_t n - \ell_t}{\mathcal{P}}\right) = -\left(\frac{-24\widetilde{\ell}_t}{\mathcal{P}}\right)} p(n) q^{\ell_t n - \frac{\ell_t}{24}} \tag{3.28}$$

is a weakly holomorphic modular form of weight $-1/2$ on $\Gamma_1(576\widetilde{\ell}_t\mathcal{P}^4)$. In particular, all of these forms are modular with respect to $\Gamma_1(576 f_t^2 \widetilde{\ell}_t \mathcal{P}^4)$.

Since $Q \nmid 576 f_t^2 \widetilde{\ell}_t \mathcal{P}^4$, a result of Treneer (see Theorem 3.1 of [32]), generalizing earlier observations of Ahlgren and Ono [2, 25], implies that there is a sufficiently large integer m for which

$$\sum_{Q \nmid n} a(r,t,\mathcal{P};Q^m n)q^n,$$

for all $0 \leq r < t$, and

$$\sum_{Q \nmid n} p(t,\mathcal{P};Q^m n)q^n$$

are all congruent modulo Q^j to forms in the graded ring of half-integral weight cusp forms with algebraic integer coefficients on $\Gamma_1(576 f_t^2 \widetilde{\ell}_t \mathcal{P}^4 Q^2)$.

The system of simultaneous congruences (3.26), in the case of these forms, guarantees that a positive proportion of primes L have the property that these forms modulo Q^j are annihilated by the index L^2 half-integral weight Hecke operators. Theorem 3.6 now follows *mutatis mutandis* as in the proof of Theorem 1 of [25]. □

Two remarks.
1) The simultaneous system (3.26) of congruences follows from a straightforward generalization of a classical observation of Serre (see Section 6 of [30]).
2) Treneer states her result for weakly holomorphic modular forms on $\Gamma_0(4N)$ with Nebentypus. We are using a straightforward extension of her result to $\Gamma_1(4N)$.

3.3. **The Andrews-Dragonette Conjecture for $f(q)$.** Rademacher famously employed the modularity of (1.1) to perfect the Hardy-Ramanujan asymptotic formula

$$p(n) \sim \frac{1}{4n\sqrt{3}} \cdot e^{\pi\sqrt{2n/3}} \tag{3.29}$$

to obtain his exact formula for $p(n)$. To state his formula, let $I_s(x)$ be the usual I-Bessel function of order s. Furthermore, if $k \geq 1$ and n are integers, then let

$$A_k(n) := \frac{1}{2}\sqrt{\frac{k}{12}} \sum_{\substack{x \pmod{24k} \\ x^2 \equiv -24n+1 \pmod{24k}}} \chi_{12}(x) \cdot e\left(\frac{x}{12k}\right), \tag{3.30}$$

where the sum runs over the residue classes modulo $24k$, and where

$$\chi_{12}(x) := \left(\frac{12}{x}\right). \tag{3.31}$$

If n is a positive integer, then one version of Rademacher's formula reads [27]

$$p(n) = 2\pi(24n-1)^{-\frac{3}{4}} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} \cdot I_{\frac{3}{2}}\left(\frac{\pi\sqrt{24n-1}}{6k}\right). \tag{3.32}$$

If $N_e(n)$ (resp. $N_o(n)$) denotes the number of partitions of n with even (resp. odd) rank, then by letting $w = -1$ in (3.2) we obtain

$$1 + \sum_{n=1}^{\infty}(N_e(n) - N_o(n))q^n = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2}. \tag{3.33}$$

In the spirit of Rademacher's work, it is natural to seek exact formulas for $N_e(n)$ and $N_o(n)$. In view of (3.32) and (3.33), since

$$p(n) = N_e(n) + N_o(n),$$

this question is equivalent to the problem of deriving exact formulas for the coefficients $\alpha(n)$ of the mock theta function

$$f(q) = 1 + \sum_{n=1}^{\infty} \alpha(n)q^n := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2}. \tag{3.34}$$

The problem of estimating the coefficients $\alpha(n)$ has a long history, one which even precedes Dyson's definition of partition ranks. Indeed, Ramanujan's last letter to Hardy already includes the claim that

$$\alpha(n) = (-1)^{n-1}\frac{\exp\left(\pi\sqrt{\frac{n}{6} - \frac{1}{144}}\right)}{2\sqrt{n - \frac{1}{24}}} + O\left(\frac{\exp\left(\frac{1}{2}\pi\sqrt{\frac{n}{6} - \frac{1}{144}}\right)}{\sqrt{n - \frac{1}{24}}}\right).$$

Typical of his writings, Ramanujan offered no proof of this claim. Dragonette proved this claim in her 1951 Ph.D. thesis [16], and Andrews [3] subsequently improved upon Dragonette's work, and he proved[3] that

(3.35)
$$\alpha(n) = \pi(24n-1)^{-\frac{1}{4}} \sum_{k=1}^{[\sqrt{n}]} \frac{(-1)^{\lfloor \frac{k+1}{2} \rfloor} A_{2k}\left(n - \frac{k(1+(-1)^k)}{4}\right)}{k} \cdot I_{\frac{1}{2}}\left(\frac{\pi\sqrt{24n-1}}{12k}\right) + O(n^{\epsilon}).$$

This result falls short of the problem of obtaining an exact formula for $\alpha(n)$. In his plenary address "Partitions: At the interface of q-series and modular forms", delivered at the Millenial Number Theory Conference at the University of Illinois in 2000, Andrews highlighted this classical problem by promoting his conjecture of 1966 (see page 456 of [3], and Section 5 of [5]) for the coefficients $\alpha(n)$.

Conjecture. (Andrews-Dragonette)
If n is a positive integer, then

$$\alpha(n) = \pi(24n-1)^{-\frac{1}{4}} \sum_{k=1}^{\infty} \frac{(-1)^{\lfloor \frac{k+1}{2} \rfloor} A_{2k}\left(n - \frac{k(1+(-1)^k)}{4}\right)}{k} \cdot I_{\frac{1}{2}}\left(\frac{\pi\sqrt{24n-1}}{12k}\right). \tag{3.36}$$

Bringmann and the author have proved [11] the following theorem.

Theorem 3.7. (Bringmann-Ono, Theorem 1.1 of [11])
The Andrews-Dragonette Conjecture is true.

Sketch of the proof. By a more precise version of Theorem 3.1, which is easily deduced from work of Zwegers [35], we have that $D\left(\frac{1}{2}; z\right)$ is a weight $1/2$ weak Maass form on $\Gamma_0(144)$ with Nebentypus character $\chi_{12} = \left(\frac{12}{\cdot}\right)$. The idea behind the proof is simple. We shall construct a Maass-Poincaré series which we shall show equals $D\left(\frac{1}{2}; z\right)$. The proof of the conjecture then follows from the fact that the formulas in the Andrews-Dragonette Conjecture can be shown to give the coefficients of this Maass-Poincaré series.

Suppose that $k \in \frac{1}{2} + \mathbb{Z}$. We define a class of Poincaré series $P_k(s; z)$. For matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)$, with $c \geq 0$, define the character $\chi(\cdot)$ by

$$\chi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) := \begin{cases} e\left(-\frac{b}{24}\right) & \text{if } c = 0, \\ i^{-1/2}(-1)^{\frac{1}{2}(c+ad+1)} e\left(-\frac{a+d}{24c} - \frac{a}{4} + \frac{3dc}{8}\right) \cdot \omega_{-d,c}^{-1} & \text{if } c > 0. \end{cases} \tag{3.37}$$

Throughout, let $z = x + iy$, and for $s \in \mathbb{C}, k \in \frac{1}{2} + \mathbb{Z}$, and $y \in \mathbb{R} \setminus \{0\}$, let

$$\mathcal{M}_s(y) := |y|^{-\frac{k}{2}} M_{\frac{k}{2}\operatorname{sgn}(y),\, s-\frac{1}{2}}(|y|), \tag{3.38}$$

[3]This is a reformulation of Theorem 5.1 of [3] using the identity $I_{\frac{1}{2}}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \cdot \sinh(z)$.

where $M_{\nu,\mu}(z)$ is the standard M-Whittaker function which is a solution to the differential equation

$$\frac{\partial^2 u}{\partial z^2} + \left(-\frac{1}{4} + \frac{\nu}{z} + \frac{\frac{1}{4}-\mu^2}{z^2}\right) u = 0. \tag{3.39}$$

Furthermore, let

$$\varphi_{s,k}(z) := \mathcal{M}_s\left(-\frac{\pi y}{6}\right) e\left(-\frac{x}{24}\right).$$

Using this notation, define the Poincaré series $P_k(s;z)$ by

$$P_k(s;z) := \frac{2}{\sqrt{\pi}} \sum_{M\in\Gamma_\infty\backslash\Gamma_0(2)} \chi(M)^{-1}(cz+d)^{-k}\varphi_{s,k}(Mz). \tag{3.40}$$

Here Γ_∞ is the subgroup of translations in $\mathrm{SL}_2(\mathbb{Z})$

$$\Gamma_\infty := \left\{\pm\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix} \ : \ n\in\mathbb{Z}\right\}.$$

The defining series is absolutely convergent for $P_k\left(1-\frac{k}{2};z\right)$ for $k<1/2$, and is conditionally convergent when $k=1/2$. We are interested in $P_{\frac{1}{2}}\left(\frac{3}{4};z\right)$, which we define by analytically continuing Fourier expansions. This argument is not straightforward (see Theorem 3.2 and Corollary 4.2 of [11]). Thanks to (3.39), as a result we find that $P_{\frac{1}{2}}\left(\frac{3}{4};24z\right)$ is a weak Maass form of weight 1/2 for $\Gamma_0(144)$ with Nebentypus χ_{12}.

After a long calculation, one can show that this Maass-Poincaré series has the Fourier expansion

$$P_{\frac{1}{2}}\left(\frac{3}{4};z\right) = \left(1-\pi^{-\frac{1}{2}}\cdot\Gamma\left(\frac{1}{2},\frac{\pi y}{6}\right)\right)\cdot q^{-\frac{1}{24}} + \sum_{n=-\infty}^{0}\gamma_y(n)q^{n-\frac{1}{24}} + \sum_{n=1}^{\infty}\beta(n)q^{n-\frac{1}{24}}, \tag{3.41}$$

where for positive integers n we have

$$\beta(n) = \pi(24n-1)^{-\frac{1}{4}}\sum_{k=1}^{\infty}\frac{(-1)^{\lfloor\frac{k+1}{2}\rfloor}A_{2k}\left(n-\frac{k(1+(-1)^k)}{4}\right)}{k}\cdot I_{\frac{1}{2}}\left(\frac{\pi\sqrt{24n-1}}{12k}\right), \tag{3.42}$$

and for non-positive integers n we have

$$\begin{aligned}\gamma_y(n) = \pi^{\frac{1}{2}}|24n-1|^{-\frac{1}{4}}\cdot\Gamma\left(\frac{1}{2},\frac{\pi|24n-1|\cdot y}{6}\right)&\\ \times\sum_{k=1}^{\infty}\frac{(-1)^{\lfloor\frac{k+1}{2}\rfloor}A_{2k}\left(n-\frac{k\left(1+(-1)^k\right)}{4}\right)}{k}\cdot J_{\frac{1}{2}}\left(\frac{\pi\sqrt{|24n-1|}}{12k}\right).&\end{aligned}$$

For convenience, we let

$$P(z) := P_{\frac{1}{2}}\left(\frac{3}{4};24z\right). \tag{3.43}$$

Canonically decompose $P(z)$ into a non-holomorphic and a holomorphic part

$$P(z) = P_{nh}(z) + P_h(z). \tag{3.44}$$

In particular, we have that

$$P_h(z) = q^{-1} + \sum_{n=1}^{\infty} \beta(n) q^{24n-1}.$$

Since $P(z)$ and $D\left(\frac{1}{2}; z\right)$ are weak Maass forms of weight $1/2$ for $\Gamma_0(144)$ with Nebentypus χ_{12}, (3.41) and (3.42) imply that the proof of the conjecture reduces to proving these forms are equal. First, one shows that

$$P_{nh}(z) = -S_1\left(\frac{1}{2}; z\right). \tag{3.45}$$

To establish this, we apply Proposition 2.1. One can show that $\xi_{\frac{1}{2}}(P(z))$ is a holomorphic modular form of weight $3/2$ on $\Gamma_0(144)$ with Nebentypus χ_{12}. Using (3.41), it can be shown that the non-zero coefficients of $\xi_{\frac{1}{2}}(P(z))$ are supported on exponents in the arithmetic progression $1 \pmod{24}$. Now we apply $\xi_{\frac{1}{2}}$ to $D\left(\frac{1}{2}; z\right)$, and we find that

$$\xi_{\frac{1}{2}}\left(D\left(\frac{1}{2}; z\right)\right) = 4\vartheta(\psi; z) = 4\sum_{n=1}^{\infty} \psi(n)\, n\, q^{n^2},$$

where

$$\psi(n) := \begin{cases} 1 & \text{if } n \equiv 1 \pmod{6}, \\ -1 & \text{if } n \equiv 5 \pmod{6}. \end{cases}$$

Obviously, its Fourier coefficients are also supported on exponents in the arithmetic progression $1 \pmod{24}$. Therefore, $\xi_{\frac{1}{2}}(P(z))$ and $\xi_{\frac{1}{2}}\left(D\left(\frac{1}{2}; z\right)\right)$ are both holomorphic modular forms of weight $3/2$ on $\Gamma_0(144)$ with Nebentypus χ_{12}. Using dimension formulas for spaces of half-integral weight modular forms and the Serre-Stark Basis Theorem, it follows that

$$\dim_{\mathbb{C}}(M_{1/2}(\Gamma_0(144), \chi_{12})) = \dim_{\mathbb{C}}(S_{1/2}(\Gamma_0(144), \chi_{12})) = 0 \tag{3.46}$$

implies that

$$\dim_{\mathbb{C}}\left(M_{3/2}\left(\Gamma_0(144), \chi_{12}\right)\right) = 24.$$

Since $\xi_{\frac{1}{2}}(P(z)), \xi_{\frac{1}{2}}\left(D\left(\frac{1}{2}; z\right)\right) \in M_{3/2}(\Gamma_0(144), \chi_{12})$ both have the property that their Fourier coefficients are supported on exponents of the form $24n + 1 \geq 1$, choose a constant c so that the coefficient of q, and hence all the coefficients up to and including q^{24}, of $\xi_{\frac{1}{2}}(P(z))$ and $c \cdot \xi_{\frac{1}{2}}\left(D\left(\frac{1}{2}; z\right)\right)$ agree. By dimensionality, this in turn implies that $\xi_{\frac{1}{2}}(P(z)) = c \cdot \xi_{\frac{1}{2}}\left(D\left(\frac{1}{2}; z\right)\right)$, and so we have that

$$P_{nh}(z) = -c \cdot S_1\left(\frac{1}{2}; z\right).$$

To establish that $c = 1$, let

$$E(z) := P(z) - c \cdot D\left(\frac{1}{2}; z\right).$$

This function is a weakly holomorphic modular form of weight 1/2 on $\Gamma_0(144)$ with Nebentypus χ_{12}. By (3.35) and Corollary 4.2 of [11], it follows that

$$E(z) = P_h(z) - c \cdot D\left(\frac{1}{2}; z\right) = (1-c)q^{-1}f(q^{24}) + \sum_{n \geq 0} A(n)q^{24n-1},$$

where $|A(n)| = O\left((24n-1)^{\frac{3}{4}+\epsilon}\right)$ for positive integers n. By work of Zwegers [35], applying the map $z \mapsto -\frac{1}{z}$ returns a non-holomorphic contribution unless $c = 1$. Since $E(z)$ does not have a non-holomorphic component, it follows that $c = 1$, which in turn proves that $P_{nh}(z) = -S_1\left(\frac{1}{2}; z\right)$.

Hence it follows that

$$\begin{aligned} P(z) - D\left(\frac{1}{2}; z\right) &= P_h(z) - q^{-1}R(-1; q^{24}) \\ &= q^{-1} + \sum_{n=1}^{\infty} \beta(n)q^{24n-1} - q^{-1}f(q^{24}) = \sum_{n=1}^{\infty} \nu(n)q^{24n-1} \end{aligned}$$

is a weakly holomorphic modular form of weight 1/2 on $\Gamma_0(144)$ with Nebentypus χ_{12}. By (3.35) and Corollary 4.2 of [11] again, it can be shown that

$$|\nu(n)| = O\left(n^{\frac{3}{4}+\epsilon}\right).$$

Therefore, $P(z) - D\left(\frac{1}{2}; z\right)$ is a holomorphic modular form. However, by (3.46), this space is trivial, and so we find that $P(z) - D\left(\frac{1}{2}; z\right) = 0$, which completes the proof. $\square$

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Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706
E-mail address: ono@math.wisc.edu