Ramanujan theta function

In mathematics, the **Ramanujan theta function** generalizes the form of the Jacobi theta functions, while capturing their general properties. In particular, the Jacobi triple product takes on a particularly elegant form when written in terms of the Ramanujan theta. The function is named after Srinivasa Ramanujan

Definition:

The Ramanujan theta function is defined as

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} \, b^{n(n-1)/2}$$

for $|ab| < 1$. The Jacobi triple product identity then takes the form

$$f(a,b) = (-a;ab)_\infty \, (-b;ab)_\infty \, (ab;ab)_\infty .$$

Here, the expression $(a;q)_n$ denotes the q-Pochhammer symbol. Identities that follow from this include

$$f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;q^2)_\infty (q^2;q^2)_\infty}{(-q^2;q^2)_\infty (q;q^2)_\infty}$$

and

$$f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}$$

and

$$f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty$$

this last being the Euler function, which is closely related to the Dedekind eta function.

Mock Theta Function:

In his last letter to Hardy, Ramanujan defined 17 Jacobi theta function-like functions $F(q)$ with $|q| < 1$ which he called "mock theta functions" (Watson 1936ab, Ramanujan 1988, pp. 127-131; Ramanujan 2000, pp. 354-355). These functions are *q*-series with exponential singularities such that the arguments terminate for some power t^N. In particular, if $f(q)$ is *not* a Jacobi theta function, then it is a mock theta function if, for each root of unity ρ, there is an approximation of the form

$$f(q) = \sum_{\mu=1}^{M} t^{\lambda_\mu} \exp\left(\sum_{\nu=-1}^{N} c_{\mu\nu} t^{\nu}\right) + O(1) \qquad (1)$$

as $t \to 0^+$ with $q = \rho e^{-t}$ (Gordon and McIntosh 2000).

If, in addition, for every root of unity ρ there are modular forms $h_j^{(\rho)}(q)$ and real numbers α_j and $1 \leq j \leq J(\rho)$ such that

$$f(q) - \sum_{j=1}^{J(\rho)} q^{\alpha_j} h_j^{(\rho)}(q) \qquad (2)$$

is bounded as q radially approaches ρ, then $f(q)$ is said to be a strong mock theta function (Gordon and McIntosh 2000).

Ramanujan found an additional three mock theta functions in his "lost notebook" which were subsequently rediscovered by Watson (1936ab). The first formula on page 15 of Ramanujan's lost notebook relates the functions which Watson calls $\rho(-q)$ and $\omega(-q)$ (equivalent to the third equation on page 63 of Watson's 1936 paper), and the last formula on page 31 of the lost notebook relates what Watson calls $\nu(-q)$ and $\omega(q^2)$ (equivalent to the fourth equation on page 63 of Watson's paper). The orders of these and Ramanujan's original 17 functions were all 3, 5, or 7.

Ramanujan's "lost notebook" also contained several mock theta functions of orders 6 and 10, which, however, were not explicitly identified as mock theta functions by Ramanujan. Their properties have now been investigated in detail (Andrews and Hickerson 1991, Choi 1999).

Unfortunately, while known identities make it clear that mock theta functions of "order" n are related to the number n, no formal definition for the order of a mock theta function is known. As a result, the term "order" must be regarded merely as a convenient label when applied to mock theta functions (Andrews and Hickerson 1991).

The complete list of mock theta functions of order 3 are

$$f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2} \qquad (3)$$

$$\phi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+q^2)(1+q^4)\cdots(1+q^{2n})} \qquad (4)$$