Good intentions are not enough

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> If you want a good piece of apple pie, you either have to learn to bake well, or else buy it from someone who can.

1. Introduction.

In 1980, in response to a mathematics education system which did not work well, NCTM proposed doing something and came up with "An Agenda For Action." See [14]. There was little reaction to this proposal, and no serious objections were raised, so NCTM took this as a first step toward the Standards that followed.

Less than ten years later, NCTM published their "Curriculum and Evaluation Standards For School Mathematics", [15], the first of three volumes which defined NCTM's view of what school mathematics should be and how it should be taught. See [16] and [17] for the other two volumes.

While there was a need to do something to improve school mathematics education, NCTM did not face up to the most critical problem, the lack of firm content knowledge of far too many teachers. There were other lacks in their program. NCTM did not look seriously at mathematics education in other countries. Mathematicians were not involved in the development of the Standards. The NCTM authors of their Standards had the strange notion that it is possible to teach conceptual understanding without developing technical skill at the same time. Instances of all of these failures and what came from them will be given below.

I shall contrast the Mathematical Association of America report of 1923, [13], with the NCTM Standards from the late 1980s and early 1990s to illustrate how the errors committeed by NCTM could have been avoided.

2. The mathematical knowledge of teachers.

About six or seven years ago, I spoke with a former high school teacher who was then working on developing a school program. I had been to a library and looked at the teacher's editions of two books. One topic was very poorly presented in each, so my question was whether many school teachers could have made any sense of what was in the teacher's editions. After having my fears confirmed, I said that the biggest failure in the NCTM Standards was their not saying that far too many teachers do not know enough mathematics and that NCTM should have proposed doing something about it. The reply was that it would have been hard for NCTM to say this. Here are the two ways I suggested this could have been done. First, say that NCTM has proposed that teachers teach in a much more indirect fashion than they had, and to do this adequately requires a much firmer knowledge of mathematics. Second, look at what U.S. teachers know, and compare this with what teachers in some other countries know. When it turns out that our teachers do not know as much, say that we had underestimated what is needed to teach well and do something about it.

The first of these reasons seems clear and hardly needs comment. However, it can be illustrated easily by looking at textbooks from 10 to 15 years ago and pointing out why they looked like they did. Many elementary school texts were printed so that each lesson was on two adjacent pages, which could be opened at the start of a lesson. When I first noticed this, I asked a mathematics educator who is also a textbook author why. He said that teachers want this, so that the full lesson for the day can be open on their desk. Publishers had been told that teachers were much more comfortable with the full lesson available for them to look at easily. To go from that to a system where students are expected to develop algorithms for arithmetic computation themselves is asking for a lot of trouble. When your knowledge is restricted to what is in the text and little more, the teacher will frequently be at a loss when students come up with an answer which is not obviously either right or wrong to the teacher. This can happen in a structured setting, and did when I observed an elementary school classroom in the early 1970s. A student gave a correct answer to a question, but it was not phrased in the same way it was in the book, so the teacher said the answer was wrong. Other children answered until one used the language of the book. The first child got a very poor message, learning that mathematics does not make sense. To their credit, NCTM does not want this to happen in classes. However, when teachers do not have adequate knowledge, it will, and will happen more frequently in the sort of setting NCTM favors, where the teaching is indirect. Lest anyone get the idea that the only alternative to what has been proposed by NCTM is to have the teacher give procedures and have students work on low level problems using procedures they do not understand, that is not the case. There are other options.

The new series of mathematics programs funded by the NSF in the wake of the NCTM Standards are a place to look to see if this inadequate knowledge of teachers is compensated for by the quality of the texts and the information given in the teacher's manuals. These series were developed in response to an explicit call for programs which would be modeled on the NCTM Standards.

Project 2061 is an education project of the American Association for the Advancement of Science. This group developed an outline of what they think all students should learn in science and mathematics, [20], [21]. The second of these books contains benchmarks. Some of these benchmarks were used to evaluate 12 sets of middle school mathematics texts and programs. The criteria based on those benchmarks were compared with those given in the NCTM Curriculum and Evaluation Standards [15], and with those in the draft of the second version of what NCTM thinks school mathematics should be, [18].

The results of this evaluation have been summarized in electronic form, [22], and a more detailed version will be published, partly in book form and partly on a CD. The results were that four programs were said to be satisfactory, all of them ones funded by the NSF from the call mentioned above. The other eight were judged not to be satisfactory.

While this evaluation did not look at each part of all of the books, teachers and students will use all of them, so it is natural to look at the highest rated program, "Connected Mathematics Project", [8], to see whether this program provides the necessary amount of information for teachers, and if all of the important topics in middle school mathematics are treated adequately.

Here is an example of a question from a unit quiz, and the answer given for teachers. This is from "Moving Straight Ahead", the fifth of eight units in seventh grade, [9 ,p. 101]. Problems 6 to 9 are linear equations such as $10 = x - 2.5$. Problem 12 is:

> Each equation in questions 6-9 is a specific case of a linear equation of the form $y = mx + b$. Find the slope and the y-intercept for the equation $10 = x - 2.5$.

The answer given in the Teacher's Guide is: "The equation $10 = x - 2.5$ is a specific case of the equation $y = x - 2.5$, which has a slope of 1 and a y-intercept of -2.5."

In addition to an answer, this Guide contains two student papers and teacher's comments on them. First, the work of two pairs of students.

Kim and Beth's work:

$$\begin{array}{c|c|c} X & 1 & 2 \\ \hline Y & -15 & -5 \end{array}$$

$$\frac{\text{Rise}}{\text{Run}} = \frac{1}{1} = 1 \text{ slope}$$

$$y = x - 2.5$$

$$Y \text{ intercept} = -2.5$$

$$Y = mX + b$$

$$10 = X - 2.5$$

$$\begin{aligned} X - 2.5 &= 10 \\ 10 + 2.5 &= 12.5 \\ 12.5 &= X \end{aligned}$$

Susy and Jeff's work:
Slope - 12.5
y-intercept - -2.5

Next, what a teacher wrote about this work.

> "Beth and Kim's work for question 12 makes it cleared how they found the slope for the given equation. Their work even suggests that they may have learned something from doing this problem. By constructing and finding a couple of values for a table related to the equation, they found the rise and run between two points and thus the slope. It appears that they could not just use the equation to give slope. The question I have as a teacher is, after finding slope as they did, do the students now see how they could have found the slope for the given equation?"
>
> "Susy and Jeff received 1 point for the correct y-intercept."

So, what is wrong with this? Everything, since the question asked is not correct. The equation $10 = x - 2.5$ is not a special case of the equation $y = mx + b$. What the authors and the teacher did was to take one point on the graph of $y = x - 2.5, y = 10$. This is just a point, and x is then 12.5. The graph of the equation $10 = x - 2.5$ is the vertical line $x = 12.5$. This line does not intersect the y-axis, so it has no y-intercept, and is vertical, so its slope is infinite. If the students have learned anything they have learned that pattern matching of a simple type will give you a good grade in a math quiz. They will have also have learned some incorrect mathematics. The first pair of students did a correct calculation for a different problem. NCTM has repeatedly said that they want students to understand why the calculation being done should be done. That part of NCTM's message was not followed here.

I was told about this problem by a parent whose child took this quiz. The marking was exactly as in the text.

This is far from the only error in these books. One expects some errors in texts, such as the following. This is from the same book, [9,p.75].

> "In 1980, the town of Rio Rancho, located on a mesa outside Santa Fe, New Mexico, was destined for obscurity. But as a result hard work by its city officials, it began adding manufacturing jobs at a fast rate. As a result, the city's population grew 239% from 1980 to 1990, making Rio Rancho the fastest-growing 'small city' in the United States. The population of Rio Rancho in 1990 was 37,000.
>
> a. What was the population of Rio Rancho in 1980?
>
> b. If the same rate of population increase continues, what will the population be in the year 2000?"

In the Teacher's Guide, part a is solved by $2.39P = 37,000$, so $P = 15,481$ people in 1980. The authors stress number sense, but do not use it here. Growth from 15,000 to 37,000 is less than 200%, so the answer given cannot be correct. They forgot to use the original population in 1980, so the correct equation to solve is $P + 2.39P = 37,000$, or $3.39P = 37,000$. They made the same mistake in solving part b.

One assumes this is carelessness, and hopes that teachers know enough to catch the error. I am well aware of this type of error, since a seventh grade teacher of mine used to make such errors frequently. These errors annoyed me greatly. However, one can understand how they arise, and I am a bit more tolerant of such errors now than as a student. With the field testing which was done, this error should have been caught. The texts list about 160 teachers who field tested the books.

There is a second problem with the answer. To give the population as $15,481$ is to state the answer more precisely than is warranted by the given data. By seventh grade students should be learning about how accurately results should be stated.

The second point mentioned above, of looking at other countries, has now been done, but not by NCTM.

The Third International Mathematics and Science Study, TIMSS for short, did more than just ask students to take an exam to see how well they did. Background information on texts, information on teachers, structure of the school system, amount of homework, and many other things were studied. In one of the books written about how U.S. students did, and what might have caused this, "A Splintered Vision", [11,p.79], the authors wrote:

> "Unfortunately, there are indications that U.S. teachers are weaker in subject matter preparation and knowledge than teachers in other countries."

Second, there is new work by Liping Ma developing on work done by Deborah Ball in the United States. Ball [2] asked some mathematics questions of elementary school teachers. One was to divide 1 3/4 by 1/2 and make up a story problem which leads to this calculation. Liping Ma was one of the people involved in a follow-up project run by Ball to ask these questions of a larger group of teachers. Ma had recently come from China, and was shocked by the low level of knowledge of most of the U.S. teachers. She felt that the teachers she had had in Shanghai, and those she had taught with in China, had a deeper understanding of the mathematics.

Ma's book "Knowing and Teaching Elementary Mathematics", [10], contains a summary of answers to this and three other questions, both from Chinese teachers and from a subset of those U.S. teachers who were interviewed. The difference in the depth of knowledge is so striking that it is obvious to all who read this book. The difference does not come from the higher mathematics courses the teachers have taken, for most of the Chinese teachers have only had an introductory course in algebra and geometry. They started normal school after grade nine, spent two or three years in this training program, and did not specialize in mathematics in normal school. What they had was a good school program in mathematics taught by teachers who knew that mathematics well.

The figures of how well the U.S. and Chinese teachers did on the division of fractions problem only tells part of the story. A number of U.S. teachers could not do the calculation correctly, and only one of 23 in the sample used made up a correct story problem, and even this correct story was flawed since the answer was 3 1/2 children. The teacher was aware of the problematic nature of the question she made up, but instead of changing the story, said that the students could figure out what the fractional child meant. This was hoping for a lot, since the teacher herself was unable to change the problem into another one which did not have this drawback.

Of the 72 Chinese teachers interviewed, all did the calculation correctly, and 65 of them made up correct story problems. As striking as these figures are, a more important difference is the deeper knowledge shown by the Chinese teachers, and their good common sense. The deeper knowledge can be illustrated by the following. One said that she did not think that division by 1/2 was a good problem to give to see if children understood division of fractions, so she made up one which was 1 3/4 divided by 4/5. Another made up three different problems dealing with sugar, said she would put them on the board, have a general class discussion of them, their similarities and differences, and then have the students make up their own problems.

Here is another example. This is on pages 20-21 in [10]. Students have been using manipulatives to aid them in learning subtraction, and they and the teacher discovered that the method they used with manipulatives and the standard algorithm were different, and the one using manipulatives mirrored what one would do in everyday life when making change. The students said that the standard way was more complicated, and did not see a reason for learning it. After further discussion, which brought out the differences, the students were still not convinced they needed to learn the standard method. The teacher suggested they save the puzzle and return to it later in the year. By the end of the year, they were doing problems with larger numbers, and the students saw a reason for learning the standard way.

This should be contrasted with some of our newer programs, where standard methods of doing arithmetic computations are not only not taught, but parents are told that they should not show their children standard methods. One such example is "Investigations in Number, Data and Space", another of the NSF funded programs which were written to conform to the NCTM Standards. There is a book, [12], which gives the philosophy behind this program. In it is the following quotation:

> "Furthermore, research studies from this country and from international comparisons have shown that students who use their own procedures do quite a bit better than students who use standard ones."

Two references are given, one to a paper by Stevenson, Lee and Stigler [27]. I knew this paper, it was the first one by Harold Stevenson that I read, so I wrote Stevenson to ask if I had missed anything in this paper or other papers of his I had read. Here is his reply:

> "I have no idea where the Mokros et al. comments come from. They certainly don't have any bearing on anything we have written. Nor can I think of other sources that would be a basis for their opinions."

Liping Ma's work supports Stevenson's comment. When Ma found that the Chinese teachers had a much deeper understanding of elementary mathematics than the U.S. teachers did, she tried to find out how this knowledge was learned. She asked the same questions of 26 Chinese students in Normal School, which in China is a two or three year program. All of them did the division of fractions correctly, and 22 out of 26 made up a correct story problem. This is 85%, not much below the 90% of the teachers who made up a correct story problem. Ma then asked these questions of 20 Chinese ninth grade students in what she said was a mediocre school in Shanghai where at most half of the students were able to pass college entrance examinations. All of these students did the calculation correctly, and 8 out of 20 made up a correct story problem. [10,pp. 125-126].

Is it possible to teach standard algorithms so that students not only learn how to do the calculations correctly, but build a foundation for later study of mathematics? Of course it is, and it should be done. What happened in the U.S. is that mathematics educators looked at a system which did not work, and tried to build one which they thought would. However, this problem was too hard for them, and they have failed to build such a system. As one small instance of this, consider division of fractions in Connected Mathematics Project. It is completely missing.

Now let us compare this recent history with recommendations made in 1923 by a committee which studied secondary school mathematics. On page 16 in [13], there is the following statement.

> "The United States is far behind Europe in the scientific and professional training required of its secondary teachers."

A full chapter is devoted to the training of teachers, with summaries of reports about this in many countries, and also summaries of many of our states and cities. This chapter was written by R.C. Archibald, who was responsible for the magnificent mathematics library at Brown Univ. He was a compulsive gatherer of information, and the chapter reflects this tendency of his. Some very wise comments on the level of mathematical knowledge are given. Here is one paragraph on page 16.

> "It will be apparent from the study of this report that a successful teacher of mathematics must not only be highly trained in his subject and have a genuine enthusiasm for it but must have also peculiar attributes of personality and above all insight of a high order into the psychology of the learning process as related to the higher mental activities."

And on the previous page there is the following statement.

> "While the greater part of this report concerns itself with the content of courses in mathematics,..., the National Committee must emphasize strongly its conviction that even more fundamental is the problem of the teacher - his qualifications and training, his

personality, skill, and enthusiasm.

The greater part of the failure of mathematics is due to poor teaching. Good teachers have in the past succeeded, and will continue to succeed, in achieving highly satisfactory results with the traditional material; poor teachers will not succeed even with the newer and better material."

3. Mathematics education in other countries.

In addition to the example given above about the mathematical knowledge of teachers in other countries, there are other things which could have been learned by looking at practice elsewhere. One important topic is the amount of aid given to teachers. Harold Stevenson once told me the following story. He learned Japanese during the Second World War, so when he is in Japanese classrooms, he can follow what is being said.

Stevenson was observing a lesson in Saporo, and whenever a student made a statement which showed lack of knowledge or understanding, the teacher asked a question which forced thought about the answer, and someone, either the student or another student, would say something which would clarify the situation. After the class was over, Stevenson asked the teacher how it was that he was able to ask the right question immediately to help clarify what was being learned. The teacher showed him a large sheet of paper which had an outline of the lesson, the problem the students would be asked to do, suggestions for questions to ask to get the lesson started, examples of errors which frequently occur when teaching this lesson, and suggestions of questions to ask when these errors occur. A new book, "The Teaching Gap", by James Stigler and James Hiebert, [28], contains a brief description of how this material is developed by Japanese teachers. However, they do not provide nearly enough information for this system to be developed here. What needs to be provided are many very explicit examples of such lessons, and quite a few examples of how they were developed. This program has existed in Japan for a moderately long time, and we need to learn from it, or spend decades developing something similar. It is foolish not to take advantage of what is available. In this direction, Harold Stevenson has something like 160 Japanese lessons, and only one mathematics educator has visited to look at them. I learned of these lessons by asking Stevenson what he had, after reading a description of one of the lessons, [23]. Others could easily have done the same, since another of these lessons was described in an article published in an NCTM journal, [24].

One thing which has finally been pointed out enough so that something might be done about it is the frequent interruptions in school classes by announcements from the office. This does not happen in Japan, where a class lesson is not interrupted except for an emergency. The pace of a coherent lesson is broken by interruptions. The fact that there has not been protests about this practice from teachers and organizations of teachers such as NCTM is an indication that they do not know that such interruptions are not common in all counties. In the university where I teach, there is no intercom in the classrooms, and if one were installed and used as often as it is in our schools, there would be a revolt from the faculty. We know this is inappropriate, and interferes with learning. NCTM should say so and work to have this practice stopped.

In addition to structural aid, it is possible to learn about interesting ways to approach mathematical topics. There has been a lot written about how to teach school mathematics to prepare students for algebra. Very little has been written here about a method which has been used in Asia for at least 45 years, and probably much longer. I described this as pictorial algebra when in Singapore in June, and the mathematics educator I was talking with said that is the phrase they use informally to describe the following method of solving ratio, fraction and proportion problems before algebra is started.

Here is an illustrated problem in a Singapore textbook used in the first half of sixth grade, [19, p.35].

Consider the following problem. Peter has two thirds as many marbles as Henry. If Peter gets eight more marbles, he will have five sixths as many marbles as Henry. How many marbles do Peter and Henry have?

The following picture is drawn to illustrate the original situation.

This is changed to the following when eight marbles have been added.

The answer is then clear, Peter had four small boxes with eight marbles in each, and Henry had six, so Peter had 32 marbles and Henry had 48. This method is introduced in fourth grade and used for simple problems. In fifth it is developed more and used on slightly more complicated problems. The problem above is from a sixth grade book. Later, in sixth grade, algebra starts, and is developed seriously in seventh grade. After algebra is developed a little, the pictorial approach is dropped, since the explicit use of letters is more flexible and powerful. However, the pictorial approach mirrors what is done later with letters. Hung-Hsi Wu told me that he learned this method of solving word problems when he was in Hong Kong, so it is at least 45 years old, and probably older. It is not in our texts. I had hoped that the draft of the new version of Standards would contain this, and had lent NCTM a complete set of Singapore texts for the writing team to use. Unfortunately, they did not include this.

In the report "The Underachieving Curriculum", [11], written about the results from the Second International Mathematics Study, the claim about why the results of our eighth students were so poor was laid on the curriculum in middle school. The lack of appropriate education of our teachers was dismissed as a cause. In reporting on TIMSS, Professor William Schmidt used the phrase, "A mile wide and an inch deep", to describe our curriculum. In Schmidt's first press conference on the TIMSS results, [25], he said that a gap in course taking by our teachers did not exist. Yes, our teachers have had as many college math courses as teachers in other countries do, but the poor preparation in school mathematics makes this irrelevant.

While there is a gap in knowledge of teachers, the curriculum in U.S. schools is also a problem. This was pointed out explicitly in the 1923 report.

On page 172 in [13], there is the following statement:

> "Everywhere algebra is introduced earlier than in the United States. In certain of the German schools some work in algebra is introduced during the sixth school year and in no country except the United States, is this introductory work postponed later than the seventh school year."

This was written in 1923, and some will say that these other countries were not educating as large a fraction of their students as we were. That is true, but it is no longer true. In a recent OECD report on high school graduation rates in almost 30 countries, only Mexico had a lower high school graduation rate than the United States does. In TIMSS, we had only 25% of our eighth grade students who had studied algebra, while in most of the other countries all or almost all of their students had studied algebra by the time they took the eighth grade TIMSS test.

4. The role of mathematicians.

In the 1923 report, mathematicians played a large role, but not an exclusive role. The committee which wrote this report was appointed by the Mathematical Association of America, then a relatively new group interested primarily in teaching mathematics in colleges and universities. This committee was chaired by J. W. Young, a mathematics professor at Dartmouth.

From the Univ. of Chicago, E. H. Moore was chosen. Moore had developed the Mathematics Department at the Univ. of Chicago from its start in 1892, and in a short time it was the best mathematics department in the United States. Moore had also given his retiring Presidential Address to the American Mathematical Society on questions of education. One of Moore's Ph.D. students was Oswald Veblen, who was also on this committee. Veblen was then at Princeton, and later would be the main force behind developing the School of Mathematics at the Institute for Advanced Study. Veblen wrote an article on geometry in [30], and he and Young wrote a very successful pair of books on projective geometry. D. E. Smith was another member of this committee. He was a distinguished historian of mathematics and had coauthored a number of school texts.

The National Council of Teachers of Mathematics was not formed until 1920, four years after the appointment of this MAA committee. Three regional groups of school math teachers, from the Middle States and Maryland, from New England, and from the Central States, were asked to appoint representatives. Later four others were appointed, the Commissioner of Secondary Education from Sacramento, CA and three more teachers.

This was a distinguished committee, and they had help from others not on the committee. The chapter on mathematics curricula in other countries was written by J. C. Brown, who had written a much more detailed report on this in 1915, [3].

Mathematicians keep up with developments in their area of mathematics no matter where they happen. This means they regularly read papers written by mathematicians who live in other countries. This international perspective was probably the reason that the report [13] contained an extensive summary of what happens in other countries.

Mathematicians were not included on the writing teams for the NCTM Standards. While mathematicians of the stature of Veblen and Moore probably could not have been found to help, there were mathematicians with interest and knowledge who could have been used. NCTM has added some mathematicians to the writing team for their new version of Standards, [18]. This is a step in the right direction.

There were probably two related reasons why mathematicians were not asked to be involved in writing the NCTM Standards. Both come from the New Math, where mathematicians were heavily involved. Serious errors were made in developing the New Math, and this is not the place, nor is there time, to go into this in detail. Some mathematicians felt they had tried to develop a good program, and when it was rejected, they turned away from school mathematics. With the failure of the New Math, many mathematics educators blamed mathematicians for the failure of something they helped support, but felt they had not developed.

While mathematicians played a major role in the development of the New Math, other mathematicians pointed out some of the errors at a fairly early stage, [1]. William Duren [6] feels that corrections would have been made but the reaction to some of the failures brought down the whole project before this could be done. I am not as certain that corrections were possible. However, mathematicians were not the only people who made serious errors in the New Math period. Here is an example.

I was a young mathematician during the development of the New Math, so was not directly involved except for a minor incident in Madison, WI. The local school system proposed adoption of a math series, and a few people in the Math. Dept. at the Univ. of Wisconsin looked at one of the books. I opened one of the books in this series and saw the following definition of addition of rational numbers, which were denoted by the ordered pair (a, b). See [29, p. 119].

$$(a, b) + (c, d) = (a + c, b + d).$$

This type of formalism has no place in school mathematics, but beyond that, it made no sense to me, since I assumed that (a, b) meant a/b. It did not. What the book was doing was not getting fractions, that had been done earlier, but was getting the negative fractions, so (a, b) meant $a - b$ where a and b were positive rational numbers. I appeared at a local School Board meeting and suggested that something was wrong with the process of textbook adoption if a book for eighth grade mathematics students could not be read by a professional mathematician. This book was coauthored by five people, three university mathematics educators and other two mathematics supervisors in school districts.

Mathematicians need to be involved in the actual development and writing of Standards, not just in being asked if they support the "Vision of the Standards", as happened for the NCTM Standards. NCTM asked the Mathematical Association of America to endorse their vision. Minutes of the meeting when this happened are available. Here is a little of what was said.

John Dossey had been President of NCTM when the committee to write their Standards was appointed. He appeared at an Mathematical Association of America Board Meeting on January 15, 1989, and asked for endorsement and support of the vision for school mathematics embodied in the NCTM Curriculum and Evaluation Standards for School Mathematics. These Standards were not available at the time, but an abbreviated form was provided as was an article on these Standards which had appeared in the Virginia Mathematics Teacher. Randall Heckmann asked about proofs, especially in Euclidean geometry. Dossey explained that there was very little decrease in the theory except in two column proofs in deference to paragraph proof writing skills and convincing arguments.

How has this played out in practice? Has geometry continued to have proofs or convincing arguments, or has this slipped through the cracks? Consider something which is pushed a lot in the current reform, graphs of linear equations. One would like an argument using similar triangles that the graph of y=mx+b is really a straight line, but one can argue that this is obvious and one wastes time trying to convince students that something which is obvious needs a proof. One can say the same about the criteria for two lines to be parallel when they are given by equations of the form $y = mx + b$. However, when it comes to perpendicular lines, it is not obvious that the condition that $y = mx + b$ and $y = nx + c$ are perpendicular is $mn = -1$. In a series of books which John Dossey helped write, "Focus on Algebra", [4], "Focus on Geometry", [7], and "Focus on Advanced Algebra", [5], the definition of perpendicular lines in the two algebra books is mn=-1. In the geometry book, there is a geometric definition of perpendicular near the start of the book. Two lines are perpendicular if the angle between them is 90 degrees. A bit later, there is the following definition.

"Two nonvertical lines are parallel if and only if their slopes are equal. Two nonvertical lines are perpendicular if and only if the product of their slopes is -1."

No connection is given between these two definitions of perpendicular. The second one should be a theorem, not a definition. It is possible that in this book of almost 900 pages there is a proof that the two definitions give the same lines as perpendicular, but I was not able to find it. I wrote the two main authors months ago asking if there is such a proof, but have not received a reply.

This series is only singled out because of Dossey's role in both the development of the 1989 Standards and this series. Many other books treat perpendicular lines in the same way.

In the NCTM Curriculum and Evaluation Standards there is a downplaying of technical skills, which is unfortunate. Most mathematicians I have talked with think that skills are a vital part of mathematics, and that without them one cannot develop the understanding that we want students to have. Here is one of a number of instances where the NCTM Standards not only did not say that a skill was necessary, but specifically said not to develop it. The topic is work with fractions.

Here is what NCTM proposed in [15,page 96].

"The mastery of a small number of basic facts with common fractions (e.g. $1/4 + 1/4 = 1/2; 3/4 + 1/2 = 1\frac{1}{4}$; and $1/2 * 1/2 = 1/4$) and with decimals (e.g. $0.1 + 0.1 = 0.2$ and $0.1 * 0.1 = 0.01$) contributes to students' readiness to learn estimation and for concept development and problem solving. This proficiency in the addition, subtraction, and multiplication of fractions and mixed numbers should be limited to those with simple denominators that can be visualized concretely or pictorially and are apt to occur in real-world settings; such computation promotes conceptual understanding of the operations. This is not to suggest, however, that valuable instruction time should be devoted to exercises like $17/24+5/18$ or $5\frac{3}{4} * 4\frac{1}{4}$, which are much harder to visualize and unlikely to occur in real-life situations. Division of fractions should be approached conceptually. An understanding of what happens when one divides by a fractional numbers less than or greater than 1 is essential."

I asked Liping Ma to comment on this. Here is her reply:

"I would like to claim some interesting and important relationship between 'basic facts with common fractions (e.g. $1/4 + 1/4 = 1/2; 3/4 + 1/2 = 1\frac{1}{4}$; and $1/2 * 1/2 = 1/4$) and with decimals' 'that can be visualized concretely' and those 'much harder to visualize and unlikely to occur in real-life situations.' In fact, without the conceptual understanding of the former, it will be unlikely for one to understand the latter. However, unless one's understanding of the former is deepened and solidified by the latter (which is not as hard as people imagine), the primary conceptual understanding is still very limited and superficial and therefore too fragile to make connections to other concepts of the subject. So, students' mathematical power will be generated from a connection of the 'basic facts' and 'abstract concepts', rather than emphasizing or ignoring either of them."

Ma's comment says nicely what I have felt all along, and what most mathematicians feel about technical skills. They need to be developed along with reasons why things work. There are times when the reasons come first, and other times when the skills are introduced and reasons developed as students become proficient. The idea that one can teach conceptual understanding without being able to do something really means that the level of the concept one asks students to learn is far too weak. As a colleague, Phil Miles, said a few years ago, he is happy that reformers speak about "conceptual" understanding since it is a sign that the topic in question is being watered down. He went on to say that "conceptual understanding of an exponential function is that they increase". At the time, I thought that his example was a good joke, but recently have decided that this is a "bad joke".

A paper appeared in a book published by NCTM in 1998, so it was read by at least one referee and probably the editor. Here is a problem from this paper. As background, the deer population in some counties in Wisconsin has been increasing.

> "Students from Boomer High recently studied a herd of 100 deer living in a nearby forest. Based on the number of female deer they were able to count, they hypothesized that the total deer population could be described by one of the following two functions:
>
> #1: $P(t) = t^2 - t + 100$ or #2: $D(t) = 5t + 100$
>
> $t =$ the number of years after the study.
>
> 1. Draw a table or graph of the deer populations represented by each function for every year over a ten year period.
> 2. Compare and contrast the two functions to describe what they predict will happen to the deer population over an extended period of time."

Here is a quote written by the authors.

> "In addition, the task is structured around significant mathematics. In particular, the task focuses on 'functions that are constructed as models of real-world problems' and emphasizes 'the connections among a problem situation, its model as a function in symbolic

> form, and the graph of that function' (NCTM 1989 [15], p. 126). The teachers who developed this task chose the two functions in order to model a population that was growing linearly and one that was growing exponentially."

The use of the word "exponential" is incorrect. The function in question does not grow exponentially, it grows quadratically. Elsewhere in this paper, student answers are given. One of the students referred to this function as a nearly exponential curve, so this student has not been taught the difference between exponential and polynomial growth, just the difference between linear and nonlinear growth, and modified the only word she knew for nonlinear growth.

One of the students did not understand the notation D(t) and P(t), and treated both as multiplication of a letter by t, so divided by t to get data and a graph. She referred to the nonlinear expression as having a squared term rather than one multiplied by 5. Even with this correct statement, the authors wrote that this student "recognized that symbolically, function 1 changes exponentially and function 2 changes linearly".

This is not the only error in the paper. The last student actually wrote a nice solution to a different problem, had correct calculations (one tiny slip in one of 20 calculations) to this other problem, and found where the two graphs she drew intersected. Since each function was divided by t, the point she found is the correct intersection point for the real problem. The authors did not seem to realize this, for they wrote:

> "It is interesting that even though Kathy has calculated incorrect values, the point of intersection of the two functions determined by her procedure is the same as that determined by using the correct values of the functions: $t = 6$."

In a private conversation with one of the authors, I have become convinced that they were aware that the incorrect graphs had to intersect at the correct point, and phrased their comment in a very poor way. They also know the difference between a function which grows exponentially and one which grows quadratically. The authors would have been spared the embarrassment of having these errors published if the paper had been read by a mathematician.

The fact that one student described in quadratic function as "almost exponential" is symptomatic of the vagueness which frequently accompanies an emphasis on "conceptual understanding", although this should not be the case with real understanding. One person I correspond with about this paper tried to explain the use of "exponential" as an informal use of the word to describe a situation where something grows faster as it gets larger. To their credit, the authors did not use this flimsy excuse to explain an error.

There are other problematic features to this problem. The functions $P(t)$ and $D(t)$ should be found by some sort of modelling based on past data. They clearly were not, since $P(t)$ decreases until $t = \frac{1}{2}$ while $D(t)$ is always increasing. The authors did not make up the problem, but used it without commenting on this artificial aspect of it.

The next NCTM Yearbook had Peter Lax on the editorial panel, and I did not see any glaring errors like in the paper mentioned above. Having mathematicians involved can make a difference.

Publishers need to have mathematicians read textbooks. There are very many examples of incorrect mathematics in our school texts. While reading books for California, I was shocked to read in a book dealing with number sense that there might be one million books in an elementary school. The author had no number sense.

There is a need to have scientists read school math books when science is treated. One final example, again from Connected Mathematics Project. This also appears in "Moving Straight Ahead", [9,page 78].

> "The graph below shows the altitude of a spaceship from 10 seconds before liftoff through 7 seconds after liftoff.
>
> a. Describe the relationship between the altitude of the spaceship and time.
>
> b. What is the slope for the part of the graph that is a straight line? What does this slope represent in this situation?"

One hopes that at least some of the students will have watched the launch of a space ship and noticed that it rises very slowly at first. There is a lot of inertia to overcome.

The last sentence is a fitting conclusion to this description of a situation which needs improvement.

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