<u>(I)FUNCTION</u>

A function is a rule that assigns each number x in one set- the domain to one and only one y in the range.

Domain	Range
-2	4
-1	1
0	0
1	1
2	4
.	.
.	.
.	.
X	x^2

Domain	Range
-3	-1
-2	1
-1	3
0	5
1	7
2	9
.	.
.	.
X	2X+5

A function is set of ordered pairs (x,y) such that each x- value is paired with one and only one y-value.

{(1,1),(2,4),(3,9),(4,16)}

1-> 1

2-> 4

3-> 9

4-> 16

{(1,2),(1,4),(1,6)}

1-> 2

1-> 4

1-> 6

Are not functions, Since the same x-value (1) is paired with different y-values(2,4 &6)

For x in the domain

We write

y=f(x)

(II) <u>The inverse of a function</u>

Find the graph the inverse of f: $y=x^3-4$

Solution:

To find the Inverse, we interchange x &y and solve for y

f: $y= x^3-4$

f^{-1} :$x=y^3-4$

$x+4=y^3$

$3\sqrt{x+4}=y$

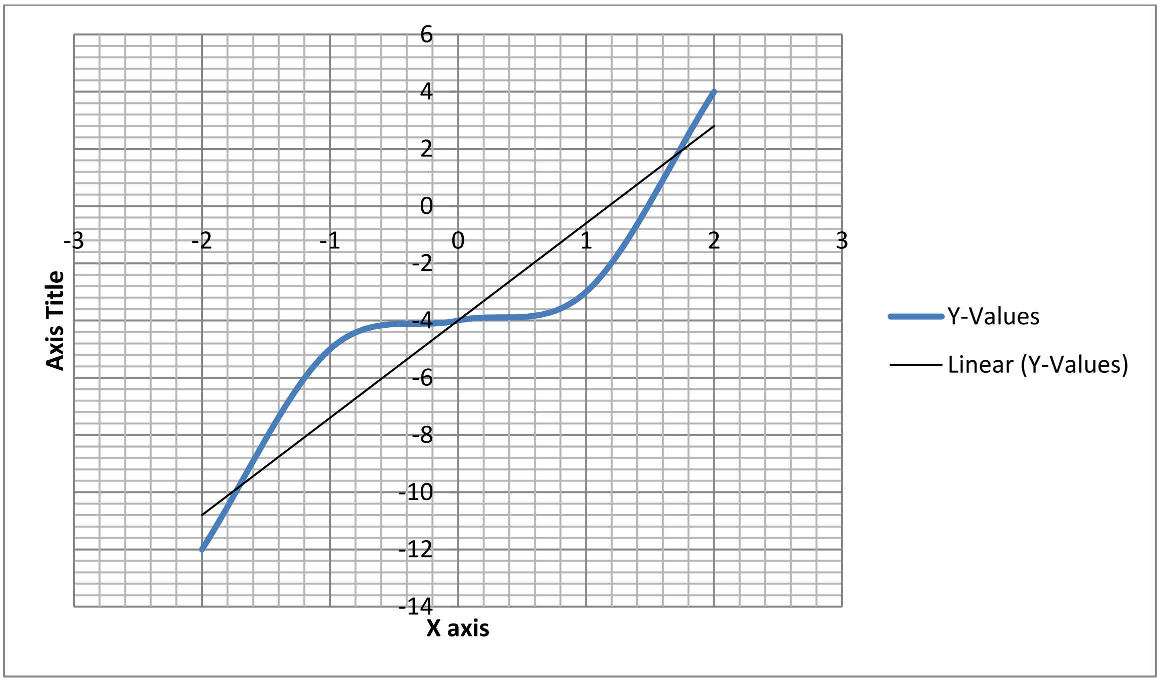
f $\qquad\qquad\qquad$ f^{-1}

x $\qquad$ y $\qquad\qquad$ x $\qquad$ y

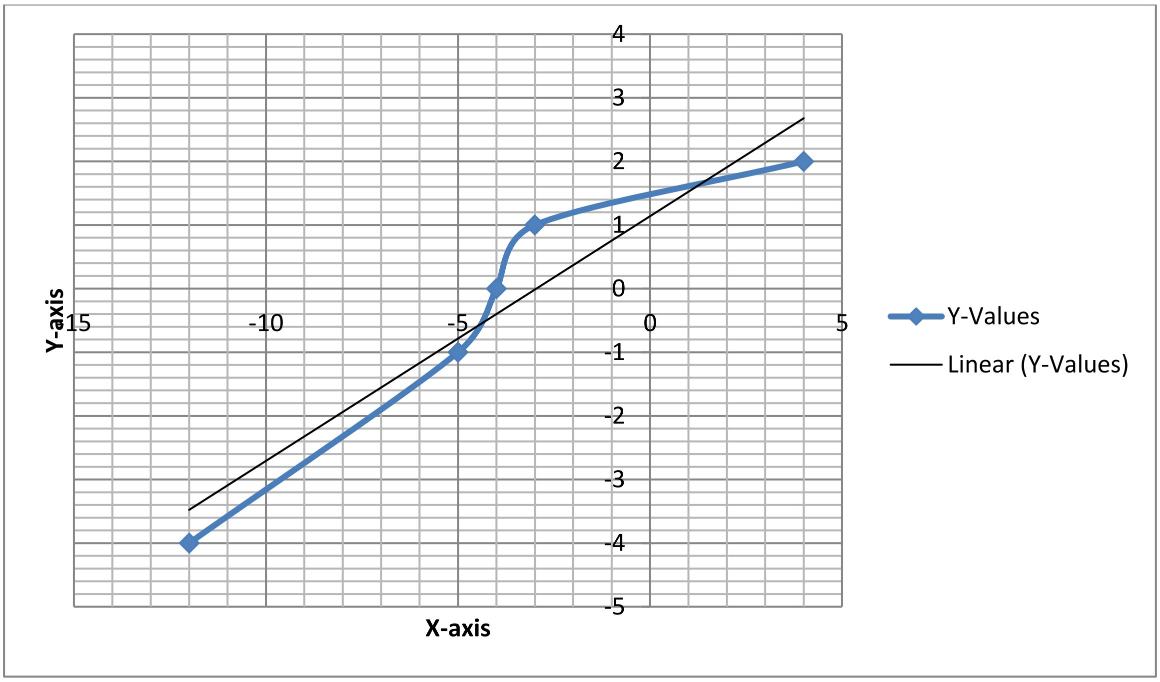
-2	-12	-12	-2
-1	-5	-5	-1
0	-4	-4	0
1	-3	-3	1
2	4	4	2

III Graphs (a) & (b)

Graph for f



Graph for f^{-1}



(IV) The graph of $y=x^2$

Table of $y=x^2$

X y

0	0
±1	1
±2	4
±3	9
±4	16

Slope through points 1 & 2 } $rise/run$

$$=\frac{y2-y1}{x2-x1}$$

$$=\frac{16-1}{4-1}=-15/3=5$$

(IV)

The graph of $y=x^2+x+1$

Table for the function $y=x^2+x+1$

x		y
0	0+0+1	1

1	1+1+1	3
2	.....	7
3	......	13
-1	....	1
-2	....	3
-1/2	....	3/4

(I) Growth and decay

The initial value problem $\frac{dx}{dy} = kx$

$X(t_o)=x_o$

Here k is a constant k occurs in either growth or decay

The rate at which certain bacteria grow is proportional to the number of bacteria present at any time.

The constant k can be determined from the solution of th D.E by using measurement of thr population at a time $t_1>t_0$.

Eq(1) also provides a model for approximating the remaining amount of the substance which is disintegrating through radioactivity.

The D.E (1) can also determine the temperature in a coolong body and also the amount of a substance remaining during a chemical reaction

(II) A culture initially has N_o number of bacteria. At t=1 hour the number of bacteria is measured to be $3/2N_o$. if the rate of growth is proportional to the number of bacteria present, Determine the time required for Bacteria to triple.

Solution: Solve the D.E

$\frac{dN}{dt} = KN$----------(i)

Subject to $N(0)=N_0$

Use the empirical condition N(1)=3/2No To determine the constant of proportionality K

Eq (i) is both separable and linear

Put into the form equation

$$\frac{dN}{dt} - KN = 0$$

By inspection note that integrating factor is e^{-kt}

Multiplying both sides of equation by this term.

This is given by $\frac{d}{dt[e-ktN]} = 0$

Integrating both sides $e^{-kt}N=C$ or $N(t)=C\,e^{-kt}$

At t=0 it follows that $N_o=Ce_0=C$

& so $N(t)=N_0e^{-kt}$

At t=1

$3/2\ N_{o=}\ N_oe^k$ or $e^k=3/2$

From this we get $K = \ln(\frac{3}{2})$

=0.4055

Thus $N(t)= N_oe^{0.4055t}$

Find the time at which bacteria get tripled

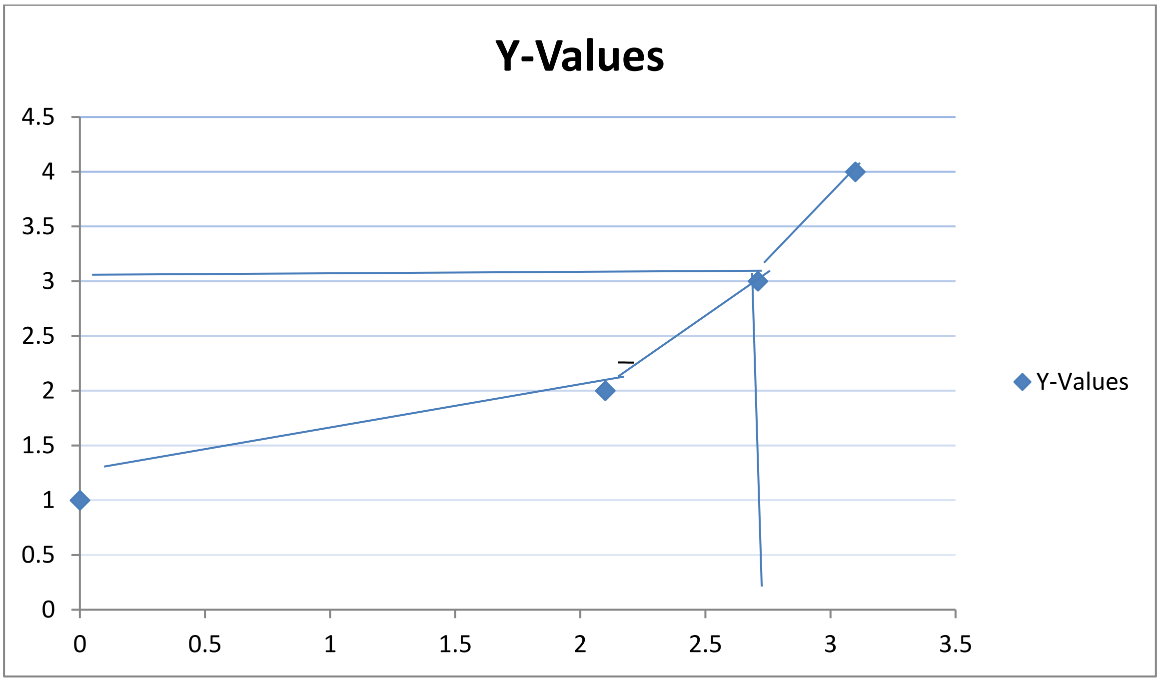
$3N_o = N_o e^{0.4055t}$

For t

It follows

$0.4055t = \ln 3$; $t = \frac{ln3}{0.4055}$

$= \approx 2.71$ hours.



We can also write the function N(t)using law of exponents

$N(t) = N_o(e^k)^t$

$= N_o(3/2)^t$

Since $e^k = 3/2$

As shown in the figure the exponential function e^{kt} increases as t increases for K>0 and decreases as t increases if k<0

Thus problems describing growth, such as population, bacteria are characterized by a positive value of K, whereas problems involving decay i.e., radioactive disintegration will yield a negative K value.

Inverse functions

Find the inverse of the fuinction f(x)=2x+1

Solution setting y=f(x)=2x+1

Solve for x as a function of y

2x=y-1

$x = \frac{y-1}{2}$

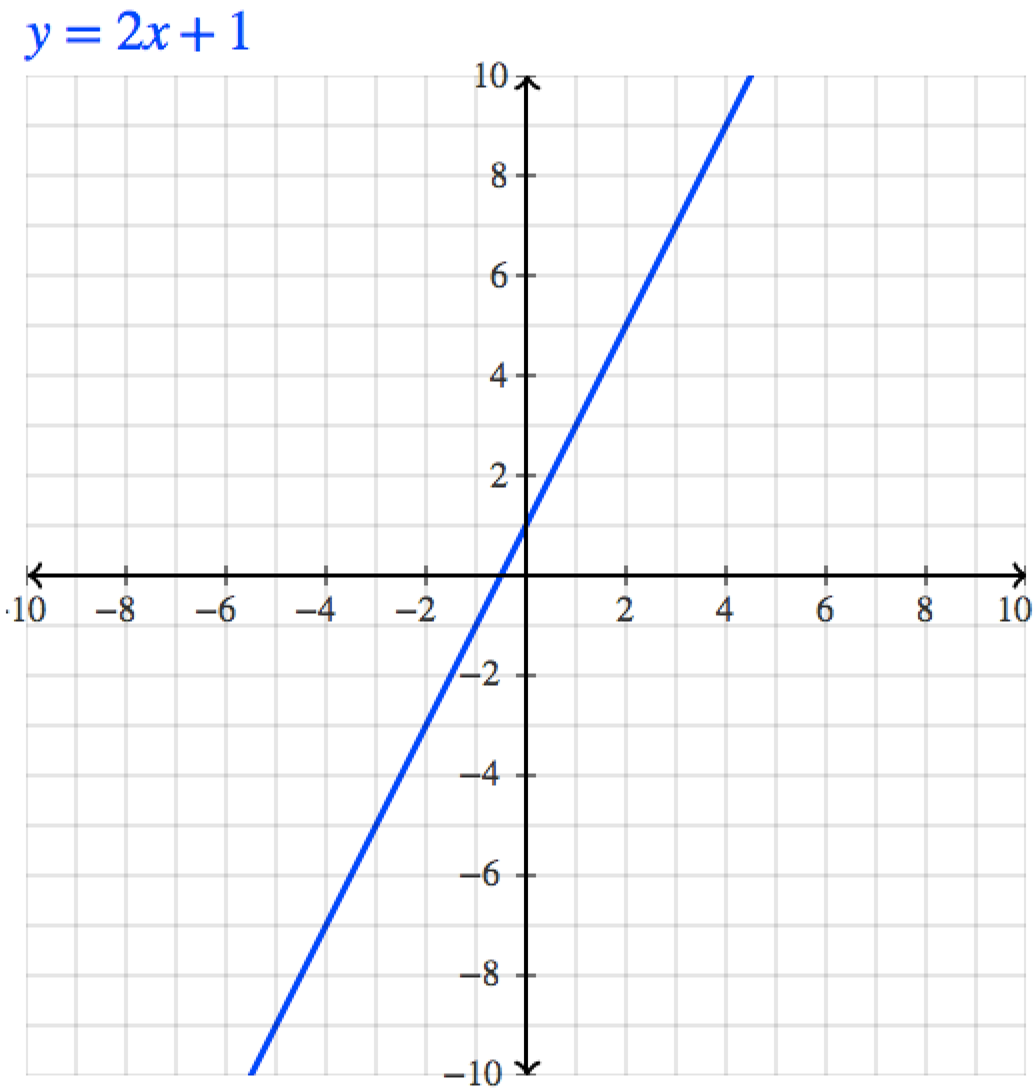
Therefore the inverse function $f^{-1}(y) = \frac{y-1}{2}$

The graph of

y=f(x)

$x=f^{-1}(y)$ are shown below

x	-2	-1	0	1	2
y	-3	-1	1	3	5



$Y=a^x$

Matrices

Definition: An array of elements in a rows and columns is called as matrix. Matrices are denoted by capital letters.

Example: A = $\begin{pmatrix} 3 & 4 \\ 7 & 5 \\ 1 & 9 \end{pmatrix}$, B = $\begin{pmatrix} 5 & 8 & 9 \\ -2 & 6 & 3 \\ 7 & -4 & 1 \end{pmatrix}$

Order of a matrix: A matrix having m rows and n columns is called a matrix of order mxn.

In the above examples A is a matrix of order 3x2 and B is a matrix of order 3x3.

In general, an mxn matrix has a following rectangular array

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

i^{th}row consists of the elements $a_{i1}, a_{i2}, \ldots\ldots\ldots\ldots a_{in}$

j^{th} column consists of the elements a_{1j} ,$a_{2j}, \ldots\ldots\ldots\ldots a_{mj}$

in generala_{ij} is an element lying in the i^{th} row and j^{th}column, we called it as the $(i,j)^{th}$ element of A. The number of elements in an mxn matrix will be equal to mn.

Types of matrices:

Row matrix: A matrix is said to be a row matrix if it has only one row.

Example: A=$\begin{pmatrix} 5 & 16 \end{pmatrix}$

Column matrix:A matrixis said to be a column matrix if it has only one column.

Example: B=$\begin{pmatrix} 12 \\ -19 \end{pmatrix}$

Square matrix: A matrix in which a number of rows are equal to the number of columns is a square matrix.

An mxn matrix is square matrix if m=n and is known as square matrix of order n.

Example: A= $\begin{pmatrix} 12 & 5 & -7 \\ 5 & 22 & 17 \\ 15 & 11 & 8 \end{pmatrix}$ is a square matrix of order 3.

In general, A=$[a_{ij}]_{mxm}$ is a square matrix of order m.

If A=$[a_{ij}]_{nxn}$ is a square matrix of order n, then elements a_{11},a_{22},…………..a_{nn} represents the diagonals of the matrix A.

If A= $\begin{pmatrix} 2 & 16 & -7 \\ 11 & 3 & 19 \\ 5 & 1 & 8 \end{pmatrix}$ Elements of the diagonal of A are 2,3,8.

Diagonal matrix:

A square matrix is diagonal matrix if all its non-diagonal elements are zero.

A= (4) B= $\begin{pmatrix} 4 & 0 \\ 0 & 7 \end{pmatrix}$ C= $\begin{pmatrix} 18 & 0 & 0 \\ 0 & 83 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ are the diagonal matrices of order 1,2,3.

Scalar matrix:

A diagonal matrix is a scalar matrix if its diagonal elements are equal and other elements are zeros

A= (7), B= $\begin{pmatrix} 12 & 0 \\ 0 & 12 \end{pmatrix}$, C= $\begin{pmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{pmatrix}$ are scalar matrices of order 1,2,3.

Identity matrix:

A square matrix in which elements in the diagonal are all one and rest are all zero is called an identity matrix and is denoted by I.

I= $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is of order 2

I = $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is of order 3

$$I = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ is of order n}$$

Zero matrix : A matrix is zero matrix or null matrix if all elements of matrix are zero s.

Ex: $A = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Triangular matrices:

1. A square matrix A= $[a_{ij}]$ is said to be upper triangular matrix if $a_{ij}=0$ for all $i>j$.

2.A square matrix A= $[a_{ij}]$ is said to be lower triangular matrix if $a_{ij}=0$ for all $i<j$.

Example:

$A = \begin{pmatrix} 10 & 3 \\ 0 & 4 \end{pmatrix}$, $A = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{pmatrix}$ are upper triangular matrices.

$A = \begin{pmatrix} 7 & 0 \\ 3 & 16 \end{pmatrix}$, $A = \begin{pmatrix} 2 & 0 & 0 \\ 3 & 5 & 0 \\ 1 & 6 & 7 \end{pmatrix}$ are lower triangular matrices.

Matrix Algebra: In the section we define arithmetic operations with matrices and look at some of their algebraic properties. As we know the entries of the matrix are called scalars. They are usually either real or complex numbers. For the most part we will be working with matrices whose entries are real numbers. Unless otherwise stated, we assume that the term scalar refers to a real number.

Definition: Two matrices A and B are said to be equal if $a_{ij} = b_{ij}$ for each i and j.

Matrix Addition:

If A =$[a_{ij}]$ and B = $[b_{ij}]$ are both mxn matrices, then the sum A+B is the mxn matrix whose ij^{th} entry is $a_{ij}+b_{ij}$ for each ordered pair(i,j).

Example: A = $\begin{pmatrix} 2 & 5 \\ -6 & 4 \end{pmatrix}$ B = $\begin{pmatrix} 7 & -4 \\ 12 & 6 \end{pmatrix}$

Then A+B = $\begin{pmatrix} 2+7 & 5-4 \\ -6+12 & 4+6 \end{pmatrix}$ = $\begin{pmatrix} 9 & 1 \\ 6 & 10 \end{pmatrix}$

If we define A-B to be A+ (-1) B, then it turns out that A-B is formed by subtracting the corresponding entry of B from each entry of A. Thus,

A-B = $\begin{pmatrix} 2-7 & 5-(-4) \\ -6-12 & 4-6 \end{pmatrix}$ = $\begin{pmatrix} -5 & 9 \\ -18 & -2 \end{pmatrix}$

Note: matrices should be added if they have same size. i.e, they have must have same number of columns and the same number of rows.

Matrix Multiplication: If A = [a_{ij}] is an mxn matrix and B = [b_{ij}] is an nxr matrix then the product AB = C= [c_{ij}] is the mxr matrix whose entries are defined by cij =

What this definition says is that to find the ijth elements of the product, you take the ith row of A and jthcolumn of B, multiply the corresponding elements pairwise, and add the resulting numbers.

Example: If A= $\begin{pmatrix} 2 & 9 \\ 4 & 7 \end{pmatrix}$,B= $\begin{pmatrix} 1 & 5 \\ -3 & 6 \end{pmatrix}$ then

AB = $\begin{pmatrix} 2 & 9 \\ 4 & 7 \end{pmatrix} . \begin{pmatrix} 1 & 5 \\ -3 & 6 \end{pmatrix}$ = $\begin{pmatrix} 2.1+9.(-3) & 2.5+9.6 \\ 4.1+7.(-3) & 4.5+7.6 \end{pmatrix}$ = $\begin{pmatrix} -16 & 64 \\ -17 & 62 \end{pmatrix}$

Scalar Multiplication: If A is a matrix and k is scalar, then kA is the matrix formed by multiplying each of the entries of A by k.

Example: A= $\begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}$ & K = 3 then KA = 3 $\begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}$ = $\begin{pmatrix} 6 & 12 \\ 18 & 24 \end{pmatrix}$

ALGEBRAIC RULES: The following theorem provides some useful rules for doing matrix arithmetic.

Theorem: Each of the following statements is valid for any scalars k and j and for any A, B and C for which the indicated operations are defined.

1. A+B = B+A

2. [A+B] +C = A+ [B+C]

3. A [B+C] = AB+AC

4. [A+B] C = AC+BC

5. [AB] C =A [BC]

6. [KJ] A = K [JA]

7. K [AB] = [KA] B =A [KB]

8. [K+J] A =KA+JA

9. K [A+B] = KA+KB

Definition: The transpose of an mxn matrix A is the nxm matrix B defined by

$b_{ij}=a_{ij}$ for j=1,……n and i=1,…..m….. The transpose of a matrix A is denoted by A^T.

Example: A = $\begin{pmatrix} 7 & 1 & 2 \\ 8 & 5 & 9 \\ 4 & 6 & 3 \end{pmatrix}$ then A^T = $\begin{pmatrix} 7 & 8 & 4 \\ 1 & 5 & 6 \\ 2 & 9 & 3 \end{pmatrix}$

Symmetric and Skew –Symmetric Matrices:

Definition: An nxn matrix A is said to be symmetric if A^T =A.

Example: $\begin{pmatrix} 7 & 1 & 4 \\ 1 & 5 & 6 \\ 4 & 6 & 3 \end{pmatrix}$

There are four basic algebraic rules involving transposes.

1. $(A^T)^T$ =A
2. $(kA)^T = kA^T$
3. If A and B are both mxn matrices, then $(A+B)^T = A^T+B^T$.
4. If A is an mxn matrix and B is an nxr matrix, then $(AB)^T = B^TA^T$.

Skew –Symmetric Matrix

An nxn matrix A is said to be skew-symmetric if A^T = -A.

Example: $\begin{pmatrix} 0 & 2 & 7 \\ -2 & 0 & 6 \\ -7 & -6 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & f & g \\ -f & 0 & h \\ -g & -h & 0 \end{pmatrix}$ are the skew-symmetric matrices.

Determinants: If A = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then the number (ad-bc) is called determinant of the square matrix A and is denoted by detA or ΔA or |A|.

Ex : $\begin{pmatrix} 5 & 7 \\ 9 & 11 \end{pmatrix}$ then ΔA = -8.

The determinant of 3x3 matrix

A = $\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$ is given by |A|= $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$

$= a_1 (b_2c_3-c_2b_3)-a_2(b_1c_3-c_1b_3)+a_3(b_1c_2-c_1b_2)$

Minors:

If in the determinant A, of the nth order, we delete the i^{th} row and j^{th} columns and form a determinant from all the elements remaining, we shall have a new det of (n-1) rows and (n-1) columns.

This new det is definedto be the minor of the elements a_{ij}. For example, if detA is a det of the fourth order,

A = $\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$ the minor of element a_{11} is denoted by M_{11} is given by

$M_{11} = \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix}$

The minor element a_{22} denoted by M_{22} is given by

$M_{22} = \begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix}$

Cofactors:

The cofactor of an element a_{ij} is the minor of the element a_{ij} multiplied with $(-1)^{i+j}$ where i+j is the sum of the i^{th} row and j^{th} column. In the above example, cofactor of a_{22} is

$a_{22} = (-1)^{2+2} M_{22}$

Adjoint of square matrix:

The transpose of matrix obtained by replacing by the elements with their cofactors is called the adjoint of a matrix.

Example: $A = \begin{pmatrix} 2 & -4 & 5 \\ -3 & 5 & 7 \\ 5 & 3 & -8 \end{pmatrix}$

$M_{11} = \begin{vmatrix} 5 & 7 \\ 3 & -8 \end{vmatrix} = -61$

$M_{12} = \begin{vmatrix} 3 & 7 \\ 5 & -8 \end{vmatrix} = -11$

$M_{13} = \begin{vmatrix} -3 & 5 \\ 5 & 3 \end{vmatrix} = -34$

$M_{21} = \begin{vmatrix} -4 & 5 \\ 3 & -8 \end{vmatrix} = 17$

$M_{22} = \begin{vmatrix} 2 & 5 \\ 5 & -8 \end{vmatrix} = -41$

$M_{23} = \begin{vmatrix} 2 & -4 \\ 5 & 3 \end{vmatrix} = 26$

$M_{31} = \begin{vmatrix} -4 & 5 \\ 5 & 7 \end{vmatrix} = -53$

$M_{32} = \begin{vmatrix} 2 & 5 \\ -3 & 7 \end{vmatrix} = 29$

$M_{33} = \begin{vmatrix} 2 & -4 \\ -3 & 5 \end{vmatrix} = -2$ then the resulting matrix of minors is

$$M = \begin{pmatrix} -61 & -11 & -34 \\ 17 & -41 & 26 \\ -53 & 29 & -2 \end{pmatrix}$$

Cofactors:
Cofactors are the signed minors. The cofactor of element aij of matrix [A] is:

$C_{ij} = (-1)^{i+j} M_{IJ}$

Therefore $C_{11}=(-1)^{1+1}M_{11} = -61$, $C_{12}=(-1)^{1+2}M_{12}=11$, $C_{13} =(-1)^{1+3}M_{13} =-34$ ....$C_{33} =(-1)^{3+3}M_{33}= -2$

The resulting matrix of cofactors is:

$$C= \begin{pmatrix} -61 & 11 & -34 \\ -17 & -41 & -26 \\ -53 & -29 & -2 \end{pmatrix}$$

Adjoint Matrix:
The adjoint matrix of [A], Adj[A] is obtained by taking the transpose of the cofactor matrix of [A].

$$\text{Adj } A = C^T = \begin{pmatrix} -61 & -17 & -53 \\ 11 & -41 & -29 \\ -34 & -26 & -2 \end{pmatrix}$$

Singular Matrix:

Definition:
A square matrix A is said to be singular if det A=0.

Example: $\begin{pmatrix} 3 & 6 \\ 2 & 4 \end{pmatrix}$

Non-singular Matrix:

Definition:
A square matrix A is non-singular if det A≠0.

Example: $\begin{pmatrix} 3 & 6 \\ 12 & 15 \end{pmatrix}$

Def: Inverse of square matrix.

A non-singular matrix A is said to have inverse if there exists another non-singular matrix B of the same order such that AB = BA = I, then B is called the inverse of A and we can write B = A^{-1}.

Note: $A^{-1} = \frac{AdjA}{\det A}$, If det A≠0.

Inverse of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Example : Find the inverse of A = $\begin{pmatrix} 2 & 4 \\ 14 & 7 \end{pmatrix}$

$$A^{-1} = \frac{1}{2.7-14.4}\begin{pmatrix} 7 & -4 \\ -14 & 2 \end{pmatrix}$$

$$= \frac{1}{-42}\begin{pmatrix} 7 & -4 \\ -14 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-7}{42} & \frac{4}{42} \\ \frac{14}{42} & \frac{2}{42} \end{pmatrix}$$

Example 2. Find the inverse of A = $\begin{pmatrix} 2 & 3 & 5 \\ 1 & 4 & 1 \\ 2 & 7 & 6 \end{pmatrix}$

Solution : we know that $A^{-1} = \frac{AdjA}{\det A}$, where Adj A = $(\text{cofactor matrix})^T$

Det A = 2(24-7)-3(6-2)+5(7-8) = 17

cofactor of element aij of matrix [A] is: $C_{ij} = (-1)^{i+j} M_{IJ}$

therefore C = $\begin{pmatrix} 17 & -4 & 1 \\ 17 & 2 & -8 \\ -17 & 3 & 5 \end{pmatrix}$ and Adj A = $[C]^T$ = $\begin{pmatrix} 17 & 17 & -17 \\ -4 & 2 & 3 \\ 1 & -8 & 5 \end{pmatrix}$

Hence $A^{-1} = \frac{AdjA}{\det A} = \frac{1}{17}\begin{pmatrix} 17 & 17 & -17 \\ -4 & 2 & 3 \\ 1 & -8 & 5 \end{pmatrix}$.

Solutions of linear equations

Consider the equations $a_1x+b_1y+c_1z = d_1$

$a_2x+b_2y+c_2z = d_2$

$a_3x+b_3y+c_3z = d_3$ these equations can be written as matrix form as

AX = B where A= $\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$; X = $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$; B = $\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$ A is called coefficient matrix.

If A is non-singular matrix $X = A^{-1} B$ gives the solution of the given equations.

Problem:

Solve the following system of linear equations usingmatrix inversion method.

x+y+z=6

2x+y+z=7

x+y+2z=9.

Sol: this may be written in the form AX =B

Where A = $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$; X = $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and B = $\begin{pmatrix} 6 \\ 7 \\ 9 \end{pmatrix}$

Det A =1(2-1)-1(4-1)+1(2-1) = -1

Cofactors of the elements are

$A_{11}=1$; $A_{12}=-3$;$A_{13}=1$

$A_{21}=-1$;$A_{22}=1$;$A_{23}=0$

$A_{31}=0$;$A_{32}=1$;$A_{33}=-1$ where A_{11} is the cofactor of the element in the first row first columns, Etc.

Adj A = $\begin{pmatrix} 1 & -1 & 0 \\ -3 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$

$A^{-1} = \dfrac{AdjA}{\det A} = \begin{pmatrix} 1 & 1 & 0 \\ -3 & -1 & -1 \\ -1 & 0 & 1 \end{pmatrix}$

$$X = A^{-1}B = \begin{pmatrix} -1 & 1 & 0 \\ 3 & -1 & -1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 6 \\ 7 \\ 9 \end{pmatrix}$$

$$= \begin{pmatrix} -6+7+0 \\ 18-7-9 \\ -6+0+9 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

∴ x = 1, y = 2, z = 3 is the required solution.

Crammer methods of solving linear equations:

This method is applied only when the matrix A is non-singular.

Consider two linear equations ax+by =p ;cx+dy =q

Express the equation in matrix form AX =B, where A= $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$; X = $\begin{pmatrix} x \\ y \end{pmatrix}$; B= $\begin{pmatrix} p \\ q \end{pmatrix}$

Construct two matrices C and D from A. C is obtained by replacing the first column of A with the column of B.

∴ C = $\begin{pmatrix} p & b \\ q & d \end{pmatrix}$

D is obtained by replacing second column of A with the Column of B.

∴ D = $\begin{pmatrix} a & p \\ c & q \end{pmatrix}$

Find the det of C and D. obtain the solution for the given set of equations

$$x = \frac{\text{detC}}{\det A}; \text{y} = \frac{\text{detD}}{\det A}.$$

Example: solve the simultaneous equations using the crammer's rule.

6x-5y = -23

3x+3y = 16

Solution : Express the equation in matrix form AX =B , where A = $\begin{pmatrix} 6 & -5 \\ 3 & 3 \end{pmatrix}$,x= $\begin{pmatrix} x \\ y \end{pmatrix}$,B = $\begin{pmatrix} -23 \\ 16 \end{pmatrix}$

Now $\det A$ = $\begin{vmatrix} 6 & -5 \\ 3 & 3 \end{vmatrix}$ = 18+15= 33.

C = $\begin{pmatrix} -23 & -5 \\ 16 & 3 \end{pmatrix}$ & $\det C$ = $\begin{vmatrix} -23 & -5 \\ 16 & 3 \end{vmatrix}$ = -23(3) +16(5)= 11

D = $\begin{pmatrix} 6 & -23 \\ 3 & 16 \end{pmatrix}$ & $\det D$ = $\begin{vmatrix} 6 & -23 \\ 3 & 16 \end{vmatrix}$ =96+69 =165

$\therefore\ x = \frac{\det C}{\det A}; y = \frac{\det D}{\det A}$

X = $\frac{11}{33}$,y= $\frac{165}{33}$

X= $\frac{1}{3}$,y = 5

Gauss-jordan Method: There is another method to solve simultaneous equations.

Consider the equations

$a_1x+b_1y+c_1z = d_1$

$a_2x+b_2y+c_2z = d_2$

$a_3x+b_3y+c_3z = d_3$ the augmented matrix is $\begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{pmatrix}$ By applying elementary row transformation, we change the above matrix in the following form is $\begin{pmatrix} 1 & 0 & 0 & p \\ 0 & 1 & 0 & q \\ 0 & 0 & 1 & r \end{pmatrix}$

Then x = p ; y =q and z =r

Example: Solve the following system by using the Gauss-Jordan elimination method.

x + y + z = 5

2x + 3y + 5z = 8

4x + 5z = 2

Solution: The augmented matrix of the system is the following

$$\begin{pmatrix} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{pmatrix}$$

We will now perform row operations until we obtain a matrix in reduced row echelon form

$$\begin{pmatrix} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{pmatrix} R_2 - 2R_1 \rightarrow \begin{pmatrix} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 4 & 0 & 5 & 2 \end{pmatrix}$$

$$R_3 - 4R_1 \rightarrow \begin{pmatrix} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & -4 & 1 & -18 \end{pmatrix}$$

$$R_3 + 4R_2 \rightarrow \begin{pmatrix} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 13 & -26 \end{pmatrix}$$

$$\frac{1}{13} R_3 \rightarrow \begin{pmatrix} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & -2 \end{pmatrix}$$

$$R_2 - 3R_3 \rightarrow \begin{pmatrix} 1 & 1 & 1 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{pmatrix}$$

$$R_1 - R_3 \rightarrow \begin{pmatrix} 1 & 1 & 0 & 7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{pmatrix}$$

$$R_1 - R_2 \rightarrow \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{pmatrix}$$

From this final matrix, we can read the solution of the system. It is $x = 3$, $y = 4$, $z = -2$

Eigenvalues and eigenvectors

Let A= (a_{ij}) be an nxn matrix of order n. det(A-λI)= 0 is called the characteristic equation where λ is variable. Roots of the characteristic equation are called the characteristic or eigenvalues or latent roots of proper values of the polynomial P (λ) = 0. The set of all eigen values of A is called spectrum of A.

Eigenvector: If is the characteristic root of a square matrix A of order n, then a non-zero vector X such that AX = λX is called a characteristic vector or eigenvector of A corresponding to the root

Example: Determine the Eigen value and Eigen vectors of A = $\begin{pmatrix} 8 & -4 \\ 2 & 2 \end{pmatrix}$.

Sol: The Eigen values are the roots of the characteristic equation λ

$$\Rightarrow \begin{vmatrix} 8-\lambda & -4 \\ 2 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (8-\lambda)(2-\lambda)+8 = 0$$

$$\Rightarrow (\lambda^2-10\lambda+24) = 0$$

$$\Rightarrow (\lambda-4)(\lambda-6)=0$$

The two distinct Eigen values are λ =4,6.

Eigen vector corresponding to the Eigen value λ =4 is (A- λ I)x = 0

$$\Rightarrow \begin{pmatrix} 8-4 & -4 \\ 2 & 2-4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} 4 & -4 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$\Rightarrow 4x_1-4x_2 = 0$

$2x_1-2x_2 = 0$

$\therefore x_1 = x_2$

Hence Eigen vector corresponding to λ =4 is X = k $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Now Eigen vector corresponding to the Eigen value λ =6 is (A- λ I)x = 0

$$\Rightarrow \begin{pmatrix} 8-6 & -4 \\ 2 & 2-6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} 2 & -4 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$\Rightarrow 2x_1-4x_2 = 0$

$\therefore x_1=2x_2$

Hence Eigen vector corresponding to $\lambda = 6$ is $X = k\begin{pmatrix}1\\2\end{pmatrix}$.

Exercise

1. A = $\begin{pmatrix}2 & -4\\7 & -25\end{pmatrix}$,B= $\begin{pmatrix}12 & 17\\7 & 13\end{pmatrix}$ then find
   (i) A+B
   (ii) 3A-2B
   (iii) AB
   (iv) A^T+B^T
   (v) $(A^T)^T+B^T$
2. $M = \begin{pmatrix}10 & 20\\7 & 16\end{pmatrix}$ then find $\det M$.
3. If $J = \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}$ then find J^2, J^3.
4. If A = $\begin{pmatrix}2 & 6\\6 & 8\end{pmatrix}$, B = $\begin{pmatrix}1 & 3 & 5\\7 & 9 & 11\end{pmatrix}$ find AB and BA whichever exist.
5. Find the product of two matrices A = $\begin{pmatrix}7\\6\\3\\4\end{pmatrix}$,B= $\begin{pmatrix}2 & 4 & 1 & 6 & 8 & 7\end{pmatrix}$
6. If A = $\begin{pmatrix}1 & 0 & -4\\0 & -1 & 2\\-1 & 2 & 1\end{pmatrix}$,B = $\begin{pmatrix}5 & 8 & 4\\2 & 3 & 2\\1 & 2 & 1\end{pmatrix}$ then prove that AB = BA.
7. If A = $\begin{pmatrix}1 & 11 & 9\\5 & 16 & -3\end{pmatrix}$ verify that $(A^T)^T = A$.
8. If A = $\begin{pmatrix}1 & 4\\2 & 7\end{pmatrix}$ then show that $(A^2)^T = (A^T)^2$.
9. There are two families A& B there are 5 men, 7 women & 3 children in family A and 9 men, 3women & 4 children in B. The recommended daily allowance for calories are man: 2200, woman: 1600, children: 1900 and for proteins are man: 55gms, woman: 65gms & children: 36gms. Represents information by matrices, what is the total requirement for calories & proteins for each of the two families.
10. Find the Eigen values and Eigen vectors for the matrices
    (i) $\begin{pmatrix}-5 & 2\\2 & -2\end{pmatrix}$

(ii) $\begin{pmatrix} 6 & 8 \\ 8 & -6 \end{pmatrix}$

(iii) $\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$

(iv) $\begin{pmatrix} -3 & -7 & -5 \\ 2 & 4 & 3 \\ 1 & 2 & 2 \end{pmatrix}$

(v) $\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$

(vi)

11 .Find the Adjoint & Inverse of the following matrices.

(i) $\begin{pmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{pmatrix}$

(ii) $\begin{pmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -2 & -4 \end{pmatrix}$

12. Solve the following equations, using Cramer's rule, Gauss –Jordan method and inversion method.

(i) x + 2y + z = 2 ,3x + y - 2z=-1 ,4x - 3y -z =3

(ii) x + y + z =6 , x –y + z = 2 ,2x –y + 3z = 9.

Answers :

1. (i) $\begin{pmatrix} 14 & 13 \\ 14 & -12 \end{pmatrix}$ (ii) $\begin{pmatrix} -18 & -46 \\ 7 & -101 \end{pmatrix}$ (iii) $\begin{pmatrix} -4 & -18 \\ -91 & -206 \end{pmatrix}$ (iv) $\begin{pmatrix} 14 & 14 \\ 13 & -12 \end{pmatrix}$ (v) $\begin{pmatrix} 14 & 3 \\ 24 & -12 \end{pmatrix}$
2. 20
3. $J^2 = J^3$ $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
4. AB exists, AB = $\begin{pmatrix} 30 & 42 & 54 \\ 62 & 90 & 118 \end{pmatrix}$, BA does not exist.

5. $\begin{pmatrix} 2 & 4 & 1 & 6 & 8 & 7 \\ 12 & 24 & 6 & 36 & 48 & 42 \\ 6 & 12 & 3 & 18 & 24 & 21 \\ 8 & 16 & 4 & 24 & 32 & 28 \end{pmatrix}$

10 (i) Eigen values -1,-6: Eigen vectors $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

(ii) Eigen values 10,-10: Eigen vectors $\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

(iii) Eigen values 4,-1: Eigen vectors $\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

(iv) Eigen values 1,1,1: Eigen vector $\begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$.

(v) Eigen values 5,1,1: Eigen vectors $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$.

11. (i) Adj A = $\begin{pmatrix} 10 & -4 & -9 \\ -15 & 4 & 14 \\ 5 & -1 & -6 \end{pmatrix}$, $A^{-1} = \frac{-1}{5}\begin{pmatrix} 10 & -4 & -9 \\ -15 & 4 & 14 \\ 5 & -1 & -6 \end{pmatrix}$

(ii) Adj A = $\begin{pmatrix} -7 & 6 & -1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{pmatrix}$, $A^{-1} = \frac{1}{2}\begin{pmatrix} -7 & 6 & -1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{pmatrix}$

12.(i) x=1,y=0.z=1

(ii)x=1,y=2,z=3.

Matrices: example problem:a mixture of three foods A,B,C. The three foods A,B,C contains nutrients P,Q, R as shown in the tabular column. How to form a mixture which will have 10 ounces of P ,12 ounces of Q, 14 ounces of R

FOOD	OUNCES PER POUND OF		
	NUTRIENT P	NUTRIENT R	NUTRIENT R
A	1	2	3
B	3	4	5
C	4	5	6

Solution : let x be the number of product of food A

let y be the number of product of food B

let z be the number of product of food C then according to the problem,

$x+3y+4z = 10$

$2x+4y+5z = 12$

$3x+5y+6z = 10$

Writing the system in matrix form is given by $Ax = B$

Where

1. There are two families A and B. There are 6 men, 4 women and 2 children in family A, and 4 men , 3 women and 5 children in family B. The recommended daily allowance for calories are: man: 2000, women:1800, child:1500 and for proteins are man:50gm, women:40gm and child:30gm.

Represent the above data by matrices. Using matrix multiplication, calculate the total requirement for calories and proteins for each of the two families.

Solution:

i) Let F be the matrix representing the order of family members, and R the matrix representing components of daily allowances, viz, calories and proteins.

ii) F= 2x3 matrix, R=3x2 matrix.

$$F = \begin{matrix} M & W & C \\ 6 & 4 & 2 \\ 4 & 3 & 5 \end{matrix} \quad \text{and} \quad R = \begin{matrix} Cal & Pro \\ 2000 & 50 \\ 1800 & 40 \\ 1500 & 30 \end{matrix}$$

The total requirement of calories and proteins for each of the two families is given by the matrix multiplication.

$$FR = \begin{matrix} 6 & 4 & 2 \\ 4 & 3 & 2 \end{matrix} \begin{matrix} 2000 & 50 \\ 1800 & 40 \\ 1500 & 30 \end{matrix}$$

$$\begin{matrix} 6x2000+4x1800+2x1500 & 6x50+4x40+2x30 \\ 4x2000+3x1800+2x1500 & 4x50+3x40+2x30 \end{matrix}$$

$$\begin{matrix} 12000+7200+3000 & 300+160+60 \\ 8000+5400+3000 & 200+120+60 \end{matrix}$$

$$\begin{matrix} 22200 & 520 \\ 16400 & 380 \end{matrix}$$

Thus family A requires 22200 calories and 520gm proteins and family B requires 16400 calories and 380gm proteins.

2. There are two families A and B. There are 3 men, 4women and 1 child in family A and 1man, 1woman and 2 children in family. The recommended daily allowance for calories is man: 2000, women: 1800, child: 1600 and for proteins is: man: 60gm, women: 50gm, child: 30gm.

Represent the above information by matrices. Using matrix multiplication, calculate the total requirement of calories and proteins for each of the two families.

i) Let F be the matrix representing the order of family members, and R the matrix representing components of daily allowances, viz, calories and proteins.
ii) F= 2x3 matrix, R=3x2 matrix.

$$A= \begin{matrix} 3 & 4 & 2 \\ 1 & 1 & 2 \end{matrix} \text{ and } R= \begin{matrix} 2000 & 60 \\ 1800 & 50 \\ 1600 & 30 \end{matrix}$$

The total requirement of calories and proteins for each of the two families is given by the matrix multiplication.

$$FR= \begin{matrix} 3 & 4 & 2 \\ 1 & 1 & 2 \end{matrix} \begin{matrix} 2000 & 60 \\ 1800 & 50 \\ 1600 & 30 \end{matrix}$$

3x2000+4x1800+2x1600 3x60+4x50+2x30

1x2000+1x1800+2x1600 1x60+1x50+2x30

6000+7200+3200 180+200+60

2000+1800+3200 60+50+60

16400 440

700 170

Thus family A requires 16400 calories and 440gm proteins and family B requires 7000 calories and 170gm proteins.

LEAST SQUARE LINEAR EQUATION

For given points $(x_1,y_1),(x_2,y_2)$……………(x_n,y_n) the least square regression line can be given by f(x) = mx+b ,where m is slope and b is y- intercept. Which will minimize the sum of the squared error,which are the errors on using the regression function f(x) to the estimate the true y values.

$(y_1-f(x_1))^2+(y_2-f(x_2))^2+$…………….$+(y_n+f(x_n))^2$, where y_i, i= 1,2,….n are true values & $f(x_i)$, i= 1,2,….n are function values, $e_i = y_i - f(x_i)$ is the error approximating y_i.

Using above points we could have the following system of equations

$y_1 = (mx_1+b) + e_1$

$y_2 = (mx_2+b) + e_2$

$y_3 = (mx_3+b) + e_3$

:

$y_n = (mx_n+b)+e_n$ where e_i , i = 1,2,….n are the error using regression line.

Now set up an matrix equation Y = XA+E

where $Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$ $X = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix}$, $A = \begin{pmatrix} b \\ m \end{pmatrix}$, $E = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix}$ we now just need to solve this for A.

The solution to least squares regression equations Y = XA+E is $A = (X^TX)^{-1}X^TY$ & the sum of the square error is $SSE = E^TE$

Example : Determine the least square regression line using a matrix & find sum of the square error for the following data ,given price is Rs x and y is monthly sales.

price	sales
15	28
25	21
35	18
45	15
55	9

From the above data we have $X = \begin{pmatrix} 1 & 15 \\ 1 & 25 \\ 1 & 35 \\ 1 & 45 \\ 1 & 55 \end{pmatrix}, Y = \begin{pmatrix} 28 \\ 21 \\ 18 \\ 15 \\ 9 \end{pmatrix}$ now using $Y = XA+E$,we need to find A.

$A = (X^TX)^{-1}X^TY$

$$X^TX = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 15 & 25 & 35 & 45 & 55 \end{pmatrix} \begin{pmatrix} 1 & 15 \\ 1 & 25 \\ 1 & 35 \\ 1 & 45 \\ 1 & 55 \end{pmatrix} = \begin{pmatrix} 5 & 175 \\ 175 & 7125 \end{pmatrix}$$

$$(X^TX)^{-1} = \frac{1}{5000}\begin{pmatrix} 7125 & -175 \\ -175 & 5 \end{pmatrix}$$

$$X^TY = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 15 & 25 & 35 & 45 & 55 \end{pmatrix} \begin{pmatrix} 28 \\ 21 \\ 18 \\ 15 \\ 9 \end{pmatrix} = \begin{pmatrix} 91 \\ 2745 \end{pmatrix}$$

$$\therefore A = (X^TX)^{-1}X^TY = \frac{1}{5000}\begin{pmatrix} 7125 & -175 \\ -175 & 5 \end{pmatrix}\begin{pmatrix} 91 \\ 2745 \end{pmatrix} = \frac{1}{5000}\begin{pmatrix} 168000 \\ -2200 \end{pmatrix} \cong \begin{pmatrix} 34 \\ -0.4 \end{pmatrix}$$

$\therefore$ f(x) =34-0.4x which is the required regression line equation.

Now to find sum of the square error

price	sales	f(x)	$e_i = y_i - f(x_i)$
15	28	28	0
25	21	24	-3
35	18	20	-2
45	15	16	-1
55	9	12	-3

$$E = \begin{pmatrix} 0 \\ -3 \\ -2 \\ -1 \\ -3 \end{pmatrix}, E^T = \begin{pmatrix} 0 & -3 & -2 & -1 & -3 \end{pmatrix}$$

$$\therefore SSE = E^T E = \begin{pmatrix} 0 & -3 & -2 & -1 & -3 \end{pmatrix} \begin{pmatrix} 0 \\ -3 \\ -2 \\ -1 \\ -3 \end{pmatrix} = \begin{pmatrix} 23 \end{pmatrix}.$$

DIFFERENTIATION

Introduction:

Differentiation is the mathematics change. Solutions of many problems were reduced to computing the derivatives. Therefore, it is important to know how quickly the derivatives of these functions can be found out.

The process of finding the derivatives of a function is called Differentiation. It derives the behavior of changing quantities.For ex. falling apples, orbiting spacecrafts, growing populations, decaying radioactive materials, rising consumer prices, all can be modeled using the differentiation.

The basic idea behind derivatives is quite simple, for a given function 'f', the derivative of function denoted by 'f', tells the rate of change of 'f', the present chapter is dedicated to the rules of differentiation of functions. Here assume that the functions under considerations are defined in their domain (where the function exists). Derivative is widely applied in Geometry, Physics, Mechanics, Chemistry, Biology and other sciences.

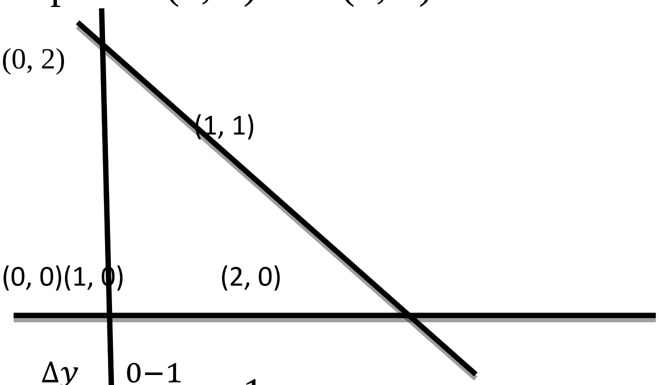
The rate of change of speed is called "*acceleration*"and speed is the rate of change of position. So acceleration is the rate of change of the position.

Derivative (or) Slope (or) Gradient of a straight line:

Take any two points on the line, the change in the x-coordinate is found by subtracting the x-coordinates of given two points and it is denoted by "Δx" (Δ is the Greek capital letter Delta), similarly change in y-coordinate is given by "Δy".

Derivative (or) Slope is $\frac{\Delta y}{\Delta x}$

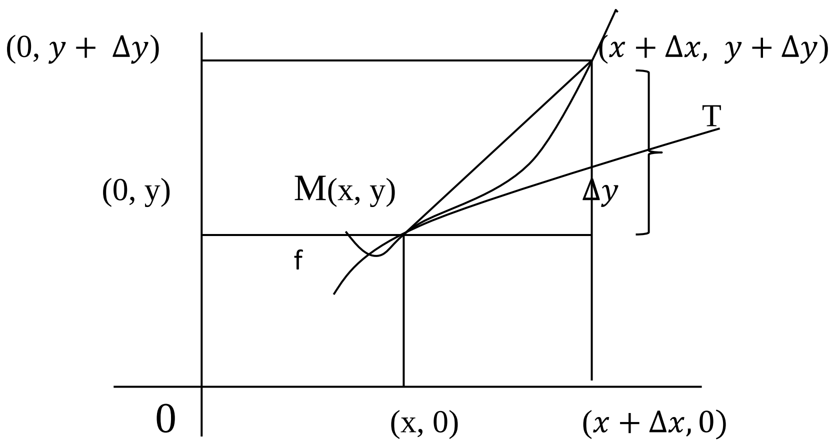
Ex: Take two points (1, 1) and (2, 0) then



slope = $\frac{\Delta y}{\Delta x} = \frac{0-1}{2-1}$ = -1

Derivative (or) Slope (or) Gradient of curves:

Let *y=f(x)* is a continuous curve.



Here MN is a chord, MT is the tangent then

$\frac{\Delta y}{\Delta x}$ = slope of the chord MN.

$= \frac{f(x+\Delta x)-f(x)}{\Delta x}$

Converting chord to tangent i.e., $\Delta x \to 0$.

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x)$$

This formula are called derivative by first principle (or) delta method.

The derivative makes it possible to study the character of a function. The greater the absolute value of the derivative, the more abruptly the function y changes as x varies and consequently, the graph of this function rises (or) falls sharply. If the derivative of some function 'y' is positive then with an increase in the value of x the function y also increase, if the derivative of the function is negative. Then this means thus as 'x' grows the function of 'y' decreases.

The derivative of a power function x^n, with respect to 'x' where n is any real constant.

Let $y = x^n$

$$\frac{\Delta y}{\Delta x} = \frac{(x+\Delta x)^n - x^n}{\Delta x} \text{taking } \Delta x \to 0 \text{ on both sides}$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x}$$

$$\frac{dy}{dx} = n.x^{n-1} \qquad (\because \lim_{x \to a} \frac{x^n - a^n}{x-a} = n.x^{n-1})$$

$$\therefore \frac{d}{dx}(x^n) = n.x^{n-1} \text{by first principle.}$$

Eg: 1. Find the derivative at x=2 of the function $f(x) = \sqrt{x}$.

Sol: $f(x) = \sqrt{x} = x^{1/2}$

$$\frac{d}{dx} f(x) = \frac{d}{dx}(x^{1/2})$$

$$= (\tfrac{1}{2}) x^{\frac{1}{2}-1} \quad (\because \frac{d}{dx}(x^n) = n.x^{n-1})$$

$$\therefore [\frac{d}{dx}(f)]_{at\ x=2} = [\tfrac{1}{2} x^{-1/2}]_{at\ x=2}$$

$$= [\tfrac{1}{2}(2)^{-1/2}]$$

$$= \frac{1}{2\sqrt{2}}.$$

2. Derivative constant function i.e.,$f(x) = c \forall x$.

Sol: $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{c-c}{\Delta x}$ $\quad (\because f(x) = c \quad \forall x)$

$= 0.$

$\therefore \frac{d}{dx}(constant) = 0.$

<u>The derivative of a sum of functions</u> : -

The derivative of finite number of sum of functions is equal to sum of the derivatives of these functions.

i.e.,$y = u + v - w$

$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} - \frac{dw}{dx}$

Eg :- If $f(x)= x^5 + x^3 + x$ find $\frac{dy}{dx}$.

Sol: $y = x^5 + x^3 + x$

$\frac{d}{dx}(y) = \frac{d}{dx}(x^5) + \frac{d}{dx}(x^3) + \frac{d}{dx}(x^1)$

$= 5x^4 + 3x^2 + 1$

<u>Product rule (or) The derivative of a product of functions</u> :-

The derivative of a product of two differentiable functions is equal to the product of first function into derivative of second plus the product of second function by the derivative of first,

i.e., $y = u.v$

$\frac{dy}{dx} = u.\frac{dv}{dx} + v.\frac{du}{dx}$

If $y = u.v.w$

$$\frac{dy}{dx} = u.v\ \frac{dw}{dx} + u.\frac{dv}{dx}\ w + \frac{du}{dx}\ v.w$$

Eg :-If $y = x^2 e^x$, find $\frac{dy}{dx}$.

Sol: $\frac{dy}{dx} = x^2 \frac{d}{dx}(e^x) + e^x \frac{d}{dx}(x^2)$

$= x^2 e^x + e^x(2x)$

$= e^x(x^2 + 2x)$

Derivative of Quotient of two functions is given by the following Quotientrule (Whenever the denominator is non-zero).

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Eg :-If $f(x) = \frac{x^2+1}{x^2-1}$. Find $f'(x)$.

Sol: $f'(x) = \frac{(x^2-1)\frac{d}{dx}(x^2+1) - (x^2+1)\frac{d}{dx}(x^2-1)}{(x^2-1)^2}$

$= \frac{(x^2-1)(2x) - (x^2+1)(2x)}{(x^2-1)^2}$ $= \frac{2x^3-2x-2x^3-2x}{(x^2-1)^2}$

$= \frac{-4x}{(x^2-1)^2}$

The derivative of trigonometric functions :-

The derivative of $Sin\ x$ with respect to x:-

$$\frac{\Delta y}{\Delta x} = \frac{Sin(x+\Delta x) - Sin(x)}{\Delta x}$$

$$\frac{\Delta y}{\Delta x} = \frac{2Sin\left(\frac{x+\Delta x-x}{2}\right)\ Cos\left(\frac{x+\Delta x+x}{2}\right)}{\Delta x}$$ $(\because SinC - Sin\ D = 2Cos(\frac{C+D}{2})\ \text{Sin}(\frac{C-D}{2}))$

$$\frac{d}{dx}(Sin\ x) = \lim_{\Delta x \to 0} \frac{2Sin\left(\frac{\Delta x}{2}\right)\ Cos\left(\ x + \frac{\Delta x}{2}\right)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2\ \ Sin\ \left(\frac{\Delta x}{2}\right)}{\left(\frac{\Delta x}{2}\right)2} Cos\left(x + \frac{\Delta x}{2}\right)$$

$$= \lim_{\Delta x \to 0} \frac{Sin\ \left(\frac{\Delta x}{2}\right)}{\left(\frac{\Delta x}{2}\right)} \lim_{\Delta x \to 0} Cos\left(x + \frac{\Delta x}{2}\right)$$

= 1. Cos $(x + 0)$ =Cos x $\therefore \frac{d}{dx} Sin\ x = Cos\ x$

<u>Table of derivatives</u> :-

f	$\frac{df}{dx}$
K	0
x	1
x^n	nx^{n-1}

f	$\frac{df}{dx}$
Cos x	-Sin x
tan x	Sec^2 x
Sec x	Sec x tan x
Cosec x	-Cosec x Cot x
Cot x	- $Cosec^2$ x

f	$\frac{df}{dx}$
e^x	e^x
e^{kx}	Ke^{kx}
a^x	$a^x \log a$
log x	$\frac{1}{x}$
log kx	$\frac{k}{x}$

Exercise:

Find the derivatives of the following functions.

(I) 1. x^{-5} 2. $x^{1/3}$ 3. 7

(II) 1. Sin $3x$ 2. e^{-7x} 3. log 5x

Solutions:

(I) 1. $\frac{d}{dx} x^{-5}$ = (-5) x^{-5-1} = (-5) x^{-6} $(\therefore \frac{d}{dx} x^n = nx^{n-1})$

2. $\frac{d}{dx} x^{1/3}$= $(\frac{1}{3})\ x^{\frac{1}{3}-1}$ = $(\frac{1}{3})\ x^{\frac{-2}{3}}$

3.$\frac{d}{dx}$ 7 = 0

(II) 1. $\frac{d}{dx}$ Sin 3x = 3. Cos 3x

2. $\frac{d}{dx} e^{-7x}$ = (-7) e^{-7x}

3. $\frac{d}{dx}$log 5x = 5($\frac{1}{5x}$) = $\frac{1}{x}$

Application of Differentiation

Reaction Rates and Rate Laws:

The area of chemistry that concerns reaction rates is called chemical kinetics. This area helps to understand the steps by which a reaction taken place. Haber process for the production of ammonia (needed for fertilizer) requires high temperatures and a catalyst iron oxide to speed up the reaction. Nitrogen dioxide (pollutant) decomposes as

$2NO_2$ (g)→ 2NO(g) +O_2 (g)---------------> 1

For any reactant A, reaction rate is defined as the change in concentration of a reactant for unit time.

$$\text{Rate}= \frac{\text{concentration A at time } t_2 - \text{concentration A at time } t_1}{t_2-t_1}$$

$$\text{Rate} = \frac{\Delta[A]}{\Delta t}$$

For the reaction given in equation 1

$$\text{Rate}= \frac{\Delta[NO_2]}{\Delta t}$$

_Δ[NO_2] has negative sign, since the concentration of [NO_2] decreases with time(substrate decreases to form the products)

Therefore with time Δ[NO_2] is a negative quatity

Rate= $\underline{-\Delta[NO_2]}$ = $\underline{d[NO_2]}$ = $K[NO_2]^n$

Δt dt

Where d indicates an infinitesimally small change

Reaction involvews a single reactant (refer eq. 1) can be generalized as

aA-------------> products

Rate= - $\frac{d[A]}{dt}$ = $K[A]^n$

when n=1(first order)

when n=2(second order)

when n=0 (zero order)

called respectively first order, second order & zero order reactions.

Example :

The decomposition of N2O5 in a gas phase was studied at constant temperature.

$2N_2O_5$ (g) ------------→ $4NO_2$ (g) + O_2 (g)

FOLLOWING RESULTS were collected.

N_2O_5 (Mol/L)	time (Sec)
0.20000	0
0.14145	50
0.10000	100
0.05000	200
0.02500	300
0.01250	400
0.00625	500

Using this data, verify that the rate law is first order in (N2O5) and calculate the value of rate constant, where the rate = -d (N_2O_5)/dt

Solution:

We can verify that the rate law is firat order in (N_2O_5) by constructing a plot of ln[N_2O_5] vs time.

The values of ln[N_2O_5] at various times are given below, and a plot of ln[N_2O_5] vs time is shown in the figure.

ln[N_2O_5]	time (Sec)
-1.6094	0
-1.956	50
-2.303	100
-2.996	200
-3.689	300
-4.382	400
-5.075	500

The plot is a straight line, conforming that the reaction is first order in N_2O_5 , since it follows the equation ln[N_2O_5] = -kt +ln[N_2O_5]$_0$

Since the reaction is first order, the slope of line equals –k . in this case

k=-(slope) = $6.93*10^{-3}$ sec^{-1}

example: Acertain first order reaction has a half life of 30mins

a. Calculate the rate constant for this reaction
b. How much time is required for this reaction to be 75% complete

Solution:

a. as we know $t_{1/2}$ =0.693/k

k= 0.693/30= $2.31*10^{-2}$ $mins^{-1}$

c. We use the integrated rate law in the form

$$\ln\left(\frac{[A]0}{[A]}\right) = kt$$

If the reaction is 75% complete, 75% of the reactant has been consumed. This leaves 25% in the original form. $\frac{[A]}{[A]0} * 100 = 25$

This means that $\frac{[A]}{[A]0} = 0.25$

And $\frac{[A]0}{[A]} = 4.0$

Therefore
$\ln\left(\frac{[A]0}{[A]}\right) = \ln(4.0) = kt$ = ($2.31*10^{-2}$ /mins)*t

And t=60mins thus it takes 60 mins for this particular reaction to reach 75% completion.

Second Order Rate Laws:

For a general reaction involving a single reactant,

aA---------→ products

which is second order in A, the rate law can be defined as

rate=-d[A]/dt=k$[A]^2$

integration of this differential rate law yields the integrated second order rate law

$$\frac{1}{[A]} = kt + 1/[A]0$$

For a second order reaction a plot of 1/[A] verses t will produce a straight line with a slope=k and the above equation shows that[A] depends on time and can be used to calculate[A] at any time t, when k &$[A]_0$ are known.

When one halflife of a second order reaction has elapsed t= t/2, [A]=$[A]_0$/2

Then

$$\frac{\frac{1}{[A]0}}{2} = kt1/2 + 1/[A]0$$

And 1/$[A]_0$= kt1/2.

Solving for t1/2 gives the expression for the half life of second order reaction

$$t1/2 = 1/k[A]0$$

Example :Butadiene reacts to form its dimer according to the equation

$2C_4H_6(g)$----------→$C_8H_{12}(g)$

The following data were collected for this reaction

C_4H_6(mol/L)	Time(±1sec)
0.01000	0
0.00625	1000
0.00476	1800
0.00370	2800
0.00313	3600
0.00270	4400
0.00241	5200
0.00208	6200

(a) Is this reaction first or second order
(b) What is the value for rate constant for the reaction
(c) What is the half life of this reaction under the conditions

Solution:

To decide whether the rate law for this reaction is first order or second order, we must see whether the plot of ln[C_4H_6] verses time is a straight line or the plot of 1/[C_4H_6] verses time is a straight line

T(s)	1/[C_4H_6]	ln[C_4H_6]
0	100	-4.605
1000	160	-5.075
1800	210	-5.348
2800	270	-5.599
3600	319	-5.767
4400	370	-5.915
5200	415	-6.028
6200	481	-6.175

The resulting plots are shown, since the ln[C4H6] verses t plot is not a straight line, the reaction is not first order, however it is second order as it shows linearity of 1/[C4H6] verses t. thuis we can write the rate law as

Rate=-d[C_4H_6]/dt=k[C_4H_6]^2

(b) for a second order reaction a plot of 1/[C_4H_6] verses t produces a straight line with a slope of k, when y=mx+c then we have y=1/[C_4H_6] and x=t.

Thus k=slope=$6.14x10^{-2}LMol^{-1}s^{-1}$

© th expression for the half life of a second order reaction is

$$t1/2 = 1/k[A]0$$

In this case k=$6.14x10^{-2}LMol^{-1}s^{-1}$

And $[A]_0$=[C_4H_6]0=0.01000M

Thus $t_{1/2}$=1/$6.14x10^{-2}LMol^{-1}s^{-1}x1.000x10^{-2}$MolL-1=$1.63x10^3$ sec

The initial concentration of C_4H_6 is halved in 1630 seconds.

Zero – order rate laws

Most reactions involving a single reactant shows either first or second order kinetics. However, some times such a reaction can be a zero order reaction. The rate law for a zero order reaction is

Rate= k$[A]^0$ = k(1) = k

For a zero order reaction the rate is constant. It does not change with concentration as it does with first and second order reaction.

The integrated rate law for a zero order reaction is

[A = kt+$[A]_0$

The expression for the half life of a zero order reaction can be obtained from the integrated rate law. By definition,(A $[A]_0/2$ when T= t/2 so

$[A]_0/2$ = -kt ½+ $[A]_0$

kt ½= $[A]_0/2$

Solving for t ½ gives kt ½

t ½ = $[A]_0/2$ k

Zero order reactions are most oftened encountered with substances such as metal surfaces or enzymes required for the reaction to occur.For example the decomposition reaction

$2N_2O(g)$ -----→ $2N_2(g)$ +$O_2(g)$

Occurs on a hot platinum surface. When the platinum surface is completely covered with N_2O molecules,an increase in the concentration of N_2O has no effect on the rate,since only those N_2O molecules on the surface can react.Under these conditions the rate is a constant because it is controlled by what happens on the platinum surface.This reaction can also occur at high temperatures with no platinum surface present , but under these conditions it is not zero order.

Radioactive Decay

Half of the radioactive substance has undergone disintegration in a period of 1500 years

1. What percentage of the radioactive substance will remain after 6000 years?
2. In how many years will one tenth of the original substance remains?
   Let x be the amount of radioactive substance present after t years
   $\frac{dx}{dt}$ represent the rate at which radioactive substance decays

Therefore $\frac{dx}{dt} = kx$

Where k is a constant of proportionality.

Since x is decreasing equation 1 should be written as $\frac{dx}{dt} = -kx$-------> 1

Letting x_0 denotes the amount initially present, we have initial condition

$x(0)=x_0$----------->2

It is given that half of the original substance disintegrates in 1500 years.

Thus x(1500)= ½ x_0------->3   (Where $x_0$denotes the amount initially present)

The differential equation 1 is separable, separating variables, Integrating and simplifying we get

$X=ce^{-kt}$ (Where c is integration constant)

Applying the initial condition (eq. 2), $x=x_0$, when $t=0$, then $c=x_0$

we obtain $x=x_0 e^{-kt}$ ----------->4

applying condition (eq. 3)

$x=1/2x_0$ when $t=1500$

eq4 we get

$1/2 x_0 = x_0 e^{-1500k}$

$(e^{-k})^{1500}=1/2$

$e^{-k}=(1/2)^{1/1500}$

$k=\ln2/1500=0.00046$

Using this (equation 4) becomes

$x= x_0 e^{-0.00046t}$

we have found $e^{-k}=(1/2)^{1/1500}$

substituting the aqbove into equation 4

$x= x_0 (e^{-k})^t$

$= x_0[(1/2)^{1/1500}]^t$

$x= x_0[(1/2)]^{t/1500}$

Question 1 asks what percentage of original substance will remain after 6000 years

$t=6000$

therefore $x= x_0[(1/2)]^{6000/1500}$

$x= x_0[(1/2)]^4=1/16 x_0$

Thus $1/16^{th}$ or 6.25% of the original substance remain after 6000 years

Question 2 is about the time required to reach one- tenth

Let $x= 1/10x_0$ (x_0=initial amount)

$1/10=(1/2)^{t/1500}$

Using logarithms

$Ln(1/10)=\ln(1/2)^{t/1500}$

= t/1500 ln(1/2)

$$\frac{t}{1500} = \frac{\ln\left(\frac{1}{10}\right)}{\ln\left(\frac{1}{2}\right)}$$

$t = 1500\frac{ln10}{ln2}$ ≈4985 years

(III) Half life

This is a measure of stabilityof radioactive substance. Half life is the time it takes for one half of the atoms in an initial amount Ao to disintegrate. The longer the half life of a substance, the more stable it is.

Half life of radium is 1700years

U238 half life is 4,500,000,000years.

Example:

A breeder reactor converts the stable uranium 238 into isotope plutonium239. After 15years it is determined that 0.043% of the initial amount A_0 of the platinum has disintegrated. Find the half life of the isotope if the rate of this isotope if the rate of disintegration is proportional to the amount remaining.

Solution:

Let A(t) denote the amount of plutonium remaining at any time

$$\frac{dA}{dt} = kA$$

$$A(0) = Ao$$

is $A(t)=A_0e^{kt}$

if 0.043% of A_0 have disegrated then 99.957% of substance remains.

Find k by solving 0.99957 $A_{0=}$ A_0e^{15k}

$e^{15k}=0.99957$

15k=ln(0.99957)

$$k = \ln(0.99957)\,/15$$

=0.00002867

$A(k)=A_oe^{-0.000002867t}$

$$t = \frac{ln2}{0.00002867}$$

=≈24,180years

Half life is the corresponding value of time for which

$A(t)=A_0/2$,

Solving for t gives $\frac{Ao}{2} = Aoe^{-0.00002867t}$

Or $\frac{1}{2} = e^{-0.00002867t}$

=0.00002867t=ln(1/2)=-ln2.

Example 2

A fossiled bone contain 1/1000 of the original amount of c-14.

Find the age of the fossil

$A(t)=A_oe^{kt}$

(c14 half life =5600years)

When t=5600years

$A(t)=A_o/2$

From the above determine k

$$\frac{Ao}{2} = Aoe^{5600k}$$

$$5600k = \ln\left(\frac{1}{2}\right) = -ln$$

$$k = -\frac{ln2}{5600}$$

=-0.00012378

$A(t)=A_oe^{-0.00012378t}$

When A(t)= $Ao/1000$

$$\frac{Ao}{1000} = A0e^{-0.00012378t}$$

-0.00012378t=ln(1/1000)

=-ln1000

$$t = \frac{ln1000}{0.00012378}$$

=55,800 years.

INTEGRATION

A function f is differentiable in an interval I, its derivative f' exists at each point of I, then a natural question arises that for the given f' at each point of I, can we obtain the function? The function that could possibly have given function as a derivative is called "*anti-derivative*"*(or) primitive* of the function. Further, the formula that gives all these anti-derivatives is called the *"indefinite integral"* of the function and such process of finding anti-derivatives is called *"integration"*.

The development of integral calculus arises out of the efforts of solving the problems of the following types.

i. The problem of finding a function whenever its derivative is given (Indefinite integrals).
ii. The problem of finding the area bounded by the graph of a function under certain conditions (Definite integrals).

The above two types together constitute the *"integral calculus"*. There is a connection, known as *"the fundamental theorem of calculus"*, between indefinite and definite integral which makes the definite integral as a practical tool for science and engineering.

INTEGRATION AS AN INVERSE PROCESS OF DIFFERENTIATON:

Integration is the inverse process of differentiation.

If $\frac{dy}{dx} = 5x^4$, what is y?

We know that if we differentiate $y = x^5$ with respect to x, we get $5x^4$, therefore we can state that the integral $5x^4$is x^5, can we? If we differentiate $y = x^5 + 1$ w.r.t x, we also get $5x^4$. Likewise, differentiating $y = x^5 + 2$ gives $5x^4$. In fact differentiating $y = x^5 + c$ where c is any constant, gives $5x^4$. So the integral of $5x^4$is $x^5 + c$.

The notation for the integration is: $\int 5x^4 dx = x^5 + c$

The $'\int'$symbol, is a form of elongated 'S', standing for Summa (Latin word for sum), indicated that what follows is to be integrated. The $'dx'$ denotes that the integration is w.r.t x, and c is the integral constant.

Standard formulae:

$\int x^n dx = \frac{x^n+1}{n+1} + c$ In particular $\int 1\, dx = \int x^0 dx = x + c$

$\int e^x\, dx = e^x + c$ $\int e^{kx}\, dx = \frac{e^{kx}}{k} + c$

$\int \frac{1}{x}\, dx = \log|x| + c$ $\int Sin\, kx\, dx = -\frac{Cos\, kx}{k} + c$

$$\int Cosx\, dx = Sin\, x + c$$

$$\int Sinx\, dx = -Cosx + c$$

$$\int Sec^2x\, dx = tanx + c$$

$$\int Cosec^2x\, dx = -Cotx + c$$

$$\int Secx.tanx\, dx = Secx + c$$

$$\int Cosecx.cotx\, dx = -Cosecx + c$$

Some properties of indefinite integrals :

1. The process of differentiation and integration are the inverse of each other in the sense of the following results.
   $\frac{d}{dx}\int f(x)dx = f(x).$

2. $\int[f(x) \pm g(x)]\, dx = \int f(x) \pm \int g(x)\, dx.$

3. For any real number K, $\int Kf(x)dx = K\int f(x)\, dx.$

Examples:

1. Write an anti-derivative for the function $Sinx + x^2$

Sol: $\int[Sinx + x^2]dx = \int Sinx\, dx + \int x^2\, dx$

$$= -Cosx + \frac{x^3}{3} + c$$

2. Find the following integrals

(i) $\int \frac{x^3-1}{x^2}\, dx$

(ii) $\int (x^{\frac{3}{2}} + 2e^x - \frac{1}{x})dx$

Sol: (i) We have $\int \frac{x^3-1}{x^2}\, dx = \int \frac{x^3}{x^2}\, dx - \int \frac{1}{x^2}\, dx$

$$= \int x\, dx - \int x^{-2}\, dx$$

$$= \frac{x^2}{2} - \frac{x^{-2+1}}{-2+1} + c \qquad = \frac{x^2}{2} + \frac{1}{x} + c$$

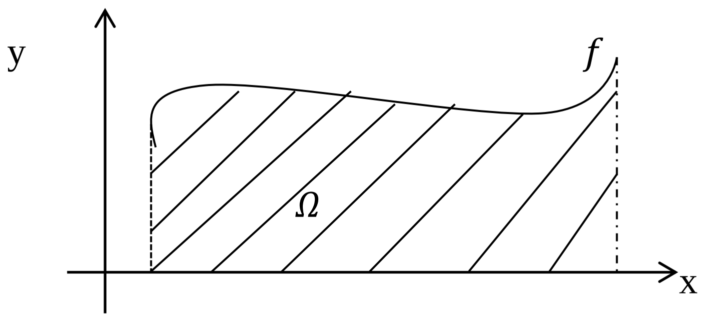
(ii) We have $\int (x^{\frac{3}{2}} + 2e^x - \frac{1}{x})dx = \int x^{3/2}\, dx + 2\int e^x dx - \int \frac{1}{x} dx$

$$= \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + 2e^x - \log|x| + c$$

$$= \frac{2}{5}x^{5/2} + 2e^x - \log|x| + c$$

Definite integral :

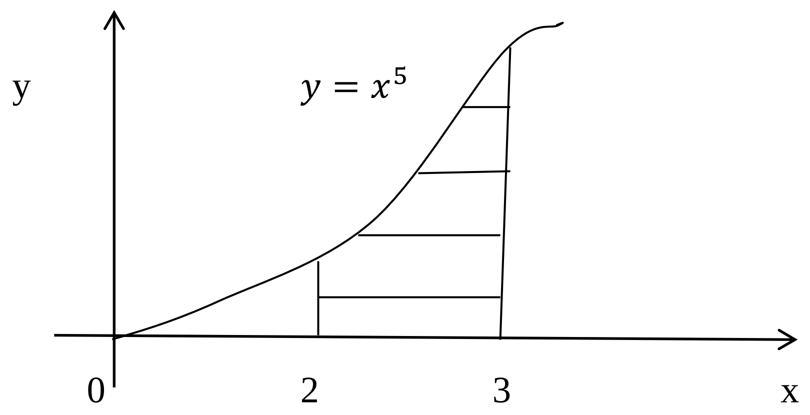
In this section, we shall study what is called definite integral of a function. The definite integral has a unique value. A definite integral has a unique value. The definite integral denoted by $\int_b^a f(x)dx$, where 'a' is called the lower limit of the integral and 'b' is called the upper limit of the integral.



In figure, you can see a region Ω bounded above by the graph of the function f, bounded below by the $x - axis$, the question before us is this; What number, if any, should be called the area of Ω?

To begin to answer this question, we shift up the intervals [a, b] into a finite number of intervals and adding all the areas we will get Ω.

For example in $\int_2^3 (5x^4)\, dx = [x^5 + c]^3$=$[3^5 + c]$ - $[2^5 + c]$= 211.



This gives us the area under the curve down to the $x - axis$ is 211. Notice that the constant of integration, c, has self-cancelled.

The area under the curve is the positive area between the curve and the $x - axis$. If the curve lies below the $x - axis$, the area between the curve and the $x - axis$ is negative.

For example, $y = x^2 - 1$ is integrated between $x = -1$ to $x = 1$ the integral is

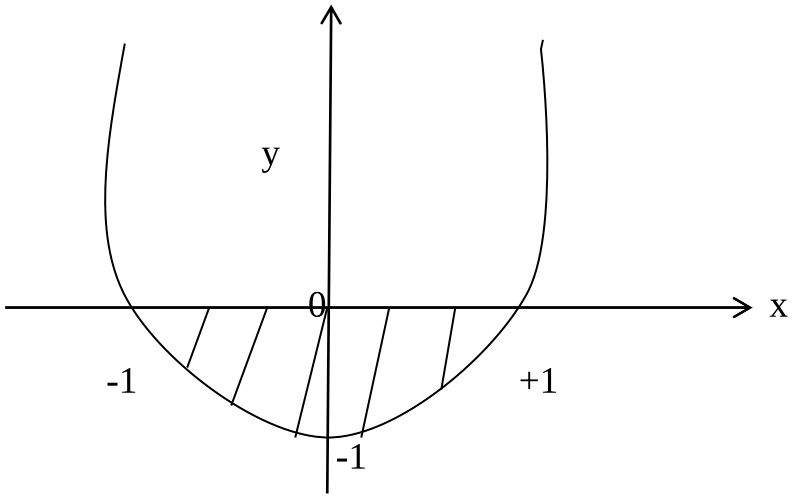
$$\int_{-1}^{1} (x^2 - 1)dx = \int_{-1}^{1} x^2 dx - \int_{-1}^{1} 1dx$$

$$= (\frac{x^3}{3})_{-1}^{1} - (x)_{-1}^{1}$$

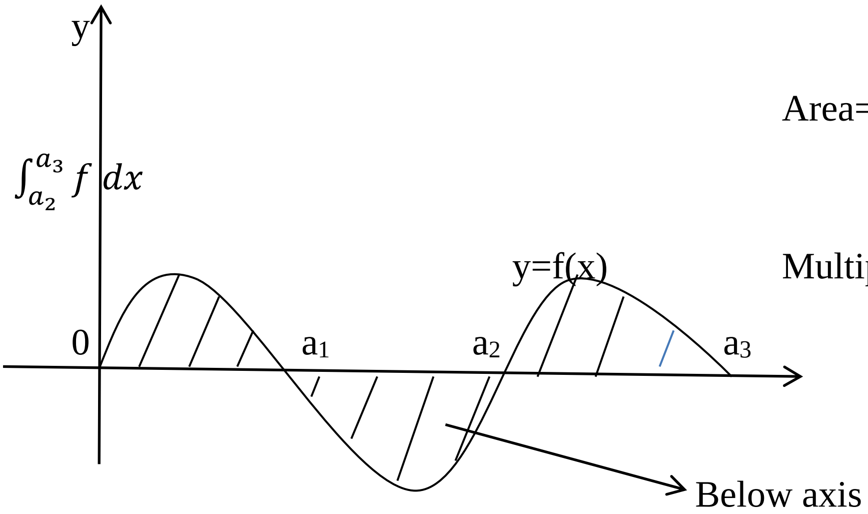
$$= \left[\frac{1}{3} - \frac{(-1)^3}{3}\right] - [\,1 - (-1)]$$

$$= \frac{2}{3} - 2 \qquad = \frac{-4}{3}$$

$$\text{Area} = -\left(\frac{-4}{3}\right) = \frac{4}{3}$$



If you wanted to treat all areas, above and below the $x - axis$, as positive you would need to multiply the area below the $x - axis$ by -1.



Area$=\int_0^{a_1} f\ dx - \int_{a_1}^{a_2} f\ dx + \int_{a_2}^{a_3} f\ dx$

Multiple with (-1)

Properties of definite integral:

1. $\int_a^a f(x)\ dx = 0$
2. $\int_a^b f(x) dx = -\int_b^a f(x)\ dx$
3. $\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$ where $c \in (a, b)$
4. $\int_a^b f(x)\ dx = \int_a^b f(t)\ \ dt$

Examples:

Evaluate the following definite integrals the limits shown

(i) $\int_3^5 (x^2 + x)\,dx = [\frac{x^3}{3} + \frac{x^2}{2}]_3^5 = (\frac{5^3}{3} + \frac{5^2}{2}) - (\frac{3^3}{3} + \frac{3^2}{2}) = 40\frac{2}{3}$

(ii) $\int_{30^0}^{60^0} Sinx\,dx = [-Cos\,x]_{30^0}^{60^0} = -[Cos\,60^0 - Cos\,30^0] =$
$-\left[\frac{1}{2} - \frac{\sqrt{3}}{2}\right] = \frac{\sqrt{3}-1}{2}$

(iii) $\int_1^2 e^{2x}\,dx = [\frac{e^{2x}}{2}]_1^2 = \frac{1}{2}[e^4 - e^2] = \frac{e^2(e^2-1)}{2}$

(iv) $\int_1^2 (x^{-1} + x^0 + x^1)\ dx = [\log|x| + x + \frac{x^2}{2}]_1^2$

$= [\log 2 + 2 + 2] - [0 + 1 + \frac{1}{2}] = \log 2 + \frac{5}{2}$

(v) $\int_0^{\pi} Cos\,x\ dx = [Sin\,x]_0^{\pi} = 0 - 0 = 0$

Exercise:

Evaluate the following definite integrals

1. $\int_2^3 2x\ dx$
2. $\int_1^e \frac{7}{x}\ dx$
3. $\int_0^{\pi/4} Cos\,2x\ dx$
4. $\int_0^{\pi/4} Sec^2x\ dx$
5. $\int_1^2 15e^{3x}\ dx$
6. $\int_{-1}^1 (x-1)(x+2)\,dx$
7. $\int_0^4 \sqrt{x}\ dx$
8. $\int_0^{\pi} \sqrt{1 + Sin\,2x}\ dx$

Answers:

1. $[2.\frac{x^2}{2}]_2^3$ $= 3^2 - 2^2$ $= 9 - 4$ $= 5$
2. $7.[\log_e x]_1^e$ $= 7[\log_e e - \log_e 1]$ $= 7[1\text{-}0]$ $=7$ $(\because \log_e e = 1)$
3. $[\frac{Sin\,2x}{2}]_0^{\pi/4}$ $= \frac{1}{2}[Sin\,2(\frac{\pi}{4}) - Sin\,2(0)] = \frac{1}{2}[1 - 0] = 1$
4. $[\tan x]_0^{\pi/4}$ $= \tan\frac{\pi}{4} - \tan 0$ $= 1 - 0$ $= 1$
5. $15\left[\frac{e^{3x}}{3}\right]_1^2$ $= 15\,[\frac{e^6 - e^3}{3}]$ $= 5\,e^3[e^3 - 1]$
6. $\int_{-1}^1 (x^2 + x - 2)dx = \left[\frac{x^3}{3} + \frac{1}{2}x^2 - 2x\right]_{-1}^1 = \frac{-10}{3}$
7. $[\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}]_0^4$ $= \frac{2}{3}[x^{\frac{3}{2}}]_0^4$ $= \frac{2}{3}[2^3 - 0] = \frac{16}{3}$
8. $\int_0^{\frac{\pi}{2}} \sqrt{Sin^2x + Cos^2x + 2Sinx\,Cosx}\ dx$ $(\because Sin^2x + Cos^2x)$

$$= \int_0^{\pi/2} \sqrt{(Sinx + Cosx)^2}\,dx$$

$$= \int_0^{\pi/2} [Sin\,x + Cos\,x]dx$$

$$= [-Cosx + Sinx]_0^{\pi/2}$$

$$= [\,0 + 1\,] - [\,-1 + 0\,]$$

$$= 2$$

Differential equations:

Equation an containing derivatives of functions are termed as differential equations, it deals with rates of change.

Eg : 1) chemical kinetics ____ concentration changing with time.

2) Quantum chemical description of bonding ___ probability density changes with position.

3) Vibrational spectroscopy___ atomic positional coordinates changing with time.

In the above example the property studied in concentration__probability density or atomic position varying with reference to time, position etc.

Separable first order differential equation in chemical kinetics:

Consider a first order rate process, with rate constant k

A---K---> B

The rate of loss of the reactant A is proportional to its concentration and is expressed in the form of the differential equation

d/dt [A] = K[A] where[A] is the concentration of the reactant at the time t.[A] is the dependent variable and t is the independent variable.

By solving equation (2) we obtain an expression which describes how the concentration of A varies with time.

Use separation variable method to rearrange equation (2)

-d[A]/[A] = K dt on integrating

-integrald[A]/[A] =integral K dt

-ln[A]=kt+c _____(3) (boundary condition in concentration of the reactant at the time t=0 is $[A]_o$)

Improve the boundary condition then c= $-\ln[A]_O$ integrated rate equation (3) becomes

$-\ln[A] = kt - \ln[A]_O$ _____________(4)

OR

$\ln[A] = -kt + \ln[A]_O$ ___________(4A)

Or

$\ln([A]/[A]_O) = -kt$ ___________(4B)

In equation (4A), [A] s an implicit function of t there is a linear relation between ln[A] and t. thus , a plot of ln[A] against t will give a straight line of slope –k and intercept $\ln[A]_O$

Rearrange equation (4B) by taking the exponential of the each side , to generate anexplicit function which shows the exponential decay of [A] as a function of the time.

$[A]/[A]_O = e^{-kt}$

Rearranging

$[A] = [A]_O e^{-kt}$

Figure below demonstrate how value of k determines the rate of loss of A

A plot $[A]/[A]_O$ against time for

(a) $K = 1\ min^{-1}$ and (b) $k = 2 min^{-1}$

An important feature of such first order reaction is the half –life $t_{1/2}$, which is the time taken for [A]to reduce to half of its initial value then for $t=t_{1/2}$, we can write

$[A]_O/2=[A]_O\ e^{-kt_{1/2}}$ this simplifies to $½ = e^{-kt_{1/2}}$ taking natural logarithms

$\ln 1/2 = -kt_{1/2}$ use the property of logarithmsln 1/a =-ln a and rewrite the above equation

$\ln 2 = kt_{1/2}$ half- life can be expressed in terms of the rate constant $t_{1/2} = \ln 2/ k$.

Uncatalyzedreactions:the reactions can be classified on the basis of dependence of the rate of reaction upon substrate concentration. For example, rate of reaction is proportional ti the concentration of asingle species raised to the first power in a first order kinetics.

For a reaction A ------>B the initial velocity of the reaction $v_o = -k[A]$ and $[At] = [A_O]e^{-kt}$____(1)

Where $[A_O]$ is the initial concentration of the reactant A, the concentration of A at time t is [At] and k is the first order rate constant of the reaction .

If the rate of reaction is proportional to the product of two concentrations, ___ the reaction is second –order kinetics.

A +B -------->C , the initial velocity of the reaction v_o = -k [A][B]

HAVE TO DRAW 3 GRAPHS

The lure weaver __ burk plot is are of several kinds of analyses that may be performed to estimate k_m and v_{max}

$1/v = k_m/v_{max}(1/[S]) +1/v_{max}$ this has the form of simple linear equation y=mx+c

$m = k_m/v_{max}$, $b = 1/v_{max}$ (refer fig).

E +S -----> ES------->E+P

The common form of equation $v_o= v_{max}[S]/k_m+[S]$.

Where v_o=initial reaction velocity

v_{max} = maximal reaction velocity

[S] = substrate concentration

k_m =Michaelis constant =$(k_2+k_3)/k_1$

$1/v_o = k_m/v_{max}\ 1/[S] + 1/v_{max}$

This reaction is the form y = mx + b gives the straight line when $1/v_o$plotted against 1/[S]

The intercept on the $1/v_o$ axis is $1/v_{max}$ and the intercept on the 1/[S] axis is $-1/k_m$.

Probability

- Probability is defined as the likelihood of an occurrence of an event.
- Probability deals with uncertainty.
- Probability maybe 0 or 1 with zero indicating impossibility of an event and one indicating certainty of an event.
- For e.g: upon tossing a coin, the result maybe head or tail.
  Similarly, the birth of a male or female is uncertain.
- This uncertainty is expressed in terms of probability.
  1. The laws of inheritance were the first major application of probability in life sciences.
  2. Applications include defects in materials, risk of a disease, chance of survival, distribution and interaction of species, etc.
  3. Laws of Mendel can also be understood by probability.
  4. Probability als o has applications in nuclear physics.

TERMS

1. **Event** :

The result of a single observation or the outcome of an experiment is called an event.

a) In a single throw of coin, the event can be head or tail.
b) In the case of birth of a child, the event can be male or female birth.
c) In a throw of a dice, the event can be any number, 1 to 6.

2. **Simple and Compound event**:

In a health study all persons with systolic blood pressure above 160 mm are called sick. This is a compound event as compared with the individual measurement of blood pressure which are called simple events. (i.e., 161,162mm…..,etc)

3. **Mutually Exclusive:**

If two events can't occur at the same time, they are known as mutually exclusive events.

4. **Equally likely:**

Events with an equal chance of occurring are known as equally likely events.

e.g., Head,Tail

Male, Female

5. **Complementary event:**

If *E* is the evnt, *E'* is the complementary event.

When a dice is thrown and if occurrence of an odd number (1,3 or 5) is an event then the complementary event would be (2,4 or 6).

The complement of an event *A* is the event that *A* does not occur. It is denoted by *A'*.

The event *A* occurs when *A'* does not occur.

$P(A) + P(A') =1$

6. **Union:**

The union of two events A and B is the set of all outcomes that are included in either A or B or both. The union is denoted as $A \cup B$.

7. **Intersection:**

The intersection of two events A and B is the set of all outcomes that are included in both A and b. It is denoted as $A \cap B$.

8. **Outcome space/Possibility space/Sample space/Exhaustive events:**

Denoted by Ω , it is the list of all possible sample events of an experiment.

e.g., For the throw of 2 coins at the same time, we observe the ordered pairs (H,H), (H,T), (T,H), (T,T).

These four events are mutually exclusive. So the outcome would be

Ω= *(H,H), (H,T), (T,H), (T,T)*

9. **Independent Events:**

Events are said to be independent if their occurrence is not influenced by the previous occurrences.

e.g., fall of a head is not influenced by previous occurrence.

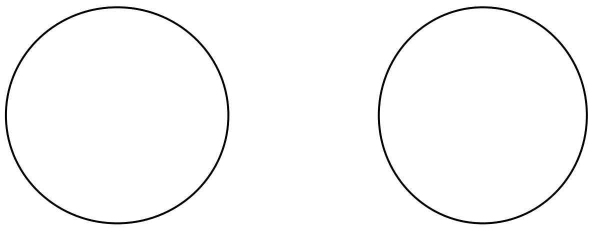
Occurrence of a male birth is not influenced by previous birth.

These events are independent of each other.

Mutually Exclusive events

1. Addition rule for mutually exclusive events:

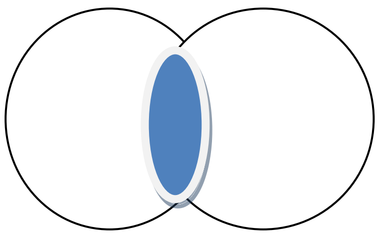
   $P(A \text{ or } B) = P(A) + P(B)$



Event A **Event B**

2. Addition rule for non-mutually exclusive events:

   $P(A \cup B) = P(A) + P(B) - P(A \cap B)$



Event A And B

Multiplication Theorem or Product rule

1. **Independent Events:**
   If A and B are independent events, then the probability of A and B is equal to the product of their respective probabilities.
   *P(A) and P(B)= P(A) * P(B)*

2. **Conditional Probability:**
   1. The conditional probability A given B(A/B) is equal to the probability of A and B (A⌒B) divided by probability B.
      P(A/B) = P(A$\frac{P(A\cap B)}{P(B)}, P(B) \neq 0$

   2. The conditional probabilities of B given A(B/A) is equal to probability of A and B(AB) divided by the probability of A.

      $$P\left(\frac{B}{A}\right) = \frac{P(A\cap B)}{P}, P(A) \neq 0$$

      (OR)

      PE_2/E_1= The probability that event E_2 will occur given that event E_1 has already occurred.
      $P(E_1 \text{ and } E_2) = P(E_1)P(E_2/E_1)$

Bayes Theorem

If E_1, E_2,……,E^n are mutually exclusive events with $P(E^i) \neq 0$, (i= 1, 2, …..,n), then for any event A which is a subset of universal set of n where i=1 E^i such that P(A)>0, we have

$$P\left(\frac{Ei}{A}\right) = P(Ei).\frac{P\left(\frac{A}{Ei}\right)}{\Sigma\, P(Ei).P\left(\frac{A}{Ei}\right)}$$

Example Problem:

1. Three urns have
a) 2 black ball, 2 red balls
b) 3 black balls, 1 red ball
c) 1 black ball, 2 red ball

One of the urns is chosen at random and a ball is drawn from it. The color of the ball is found to be red. What is the probability that it has been chosen from the first urn?

Solution:

Let A be the event that a red ball Is chosen. B is the event that the urn is chosen *i= 1,2,3 .*

P(B1) =P(B2) = P(B3) = 1/3

P(A/B1) = 2/4, P(A/B2)=1/4

P(A/B3)=2/3

P(B1/A)=?

P(B1/A)= $\dfrac{P(B1).P(\frac{A}{B1})}{\Sigma P(B1).P(\frac{A}{B1})}$

$$=\frac{\frac{1}{3}.\frac{2}{4}}{\frac{1}{3}.\frac{2}{4}+\frac{1}{3}.\frac{1}{4}+\frac{1}{3}.\frac{2}{3}}$$

$$= \frac{(0.33).(0.5)}{(0.33).(0.5)+ (0.33).(0.25)+ (0.33).(0.66)}$$

$$= \frac{0.165}{0.165+0.082+0.217}$$

$$= \frac{0.165}{0.464} = 0.355$$

Similarly,

P(B2/A) $= \dfrac{\frac{1}{3}.\frac{1}{4}}{\frac{1}{3}.\frac{2}{4}+\frac{1}{3}.\frac{1}{4}+\frac{1}{3}.\frac{2}{3}}$

$$= \frac{(0.33).(0.25)}{(0.33).(0.5)+ (0.33).(0.25)+ (0.33).(0.66)}$$

$$= \frac{0.0285}{0.165+0.082+0.217}$$

$$= \frac{0.0285}{0.464} = 0.0614$$

P(B3/A) = $= \dfrac{\frac{1}{3}.\frac{2}{3}}{\frac{1}{3}.\frac{2}{4}+\frac{1}{3}.\frac{1}{4}+\frac{1}{3}.\frac{2}{3}}$

$$= \frac{(0.33).(0.66)}{(0.33).(0.5)+ (0.33).(0.25)+ (0.33).(0.66)}$$

$$= \frac{0.217}{0.165+0.082+0.217}$$

$$= \frac{0.0217}{0.464} = 0.0467$$

Probability Distribution

Let the random variable (discrete or continuous) assume values. *x1, x2,.....xn* with probabilities p1, *p2, ...pn* then the set of probabilities associated with the random variable is called probability distribution.

➢ Binomial Distribution

1. Two outcome results from a trial, success or failure, head or tail, boy or a girl.
2. If the probability of one outcome (say head) is *p*, then the probability of the other outcome (say tail) is *q*(i.e., *1-p*).
3. Outcomes are independent. E.g., If the first child is a boy, the second child may be a boy or a girl and vice versa.
4. As the two outcomes are mutually exclusive
   P+q = 1 or q= 1-p
5. **The mean of the binomial distribution is *np*, variance =*npq*, S.D= *√npq***
6. In a binomial experiment, one is interested in the number of successes (or failures) occurring in n trials.
7. Let X represent the random variable associated with the number of successes occurring in n trials. X may take any of the discrete values 0, 1, 2 , 3 , 4, ……….,n.
8. The parameter or constant of binomial distribution are *n* and *p.*

   The probability associated with each of the possible outcomes that X may take (i.e., X= 1, 2, 3, 4, …….,n) will have frequency distribution, we call, the binomial probability distribution.

Expansions

General Expansion

$$(p + q)^n = p^n + {}^nC_1 p^{n-1}q^1 + {}^nC_2 p^{n-2} q^2 + \cdots + {}^nC_r p^{n-r} q^r + \ldots {}^nC_n q^n$$

$$(p + q)^2 = p^2 + 2pq + q^2$$

$$(p + q)^3 = p^3 + 3p^2q + 3pq^2 + q^3$$

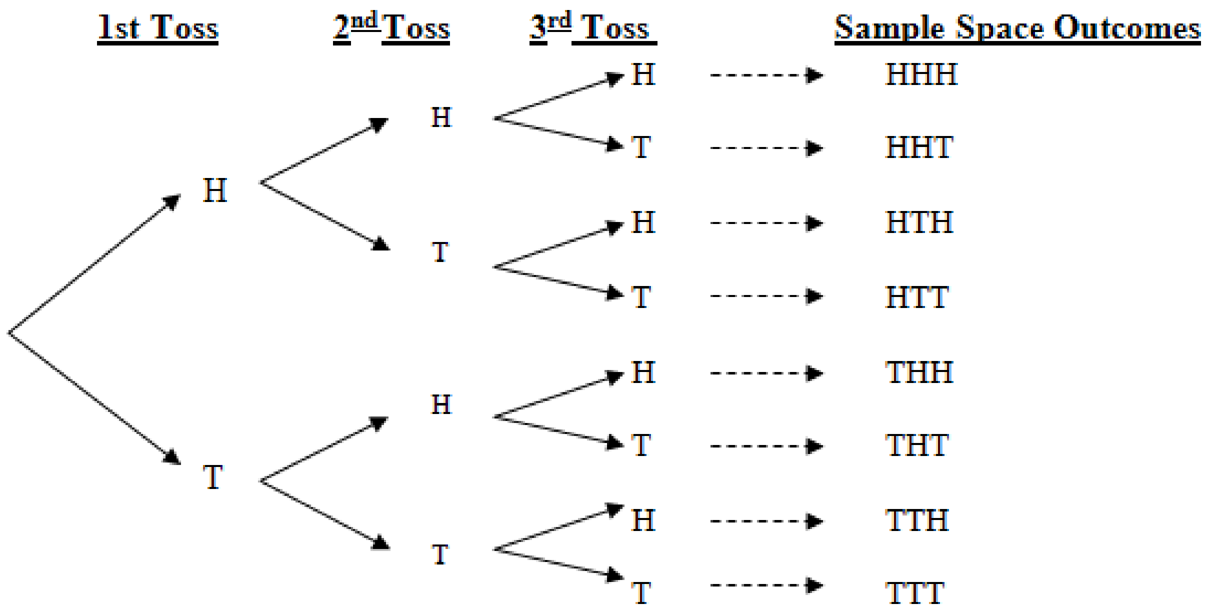
$$(p + q)^4 = p^4 + 4p^3q + 6p^2q^2 + 4pq^3 + q^4$$

$$(p + q)^5 = p^5 + 5p^4q + 10p^3q^2 + 10p^2q^3 + 5pq^4 + q^5$$

$$(p+q)^6 = p^6 + 6p^5q + 15p^4q^2 + 20p^3q^3 + 15p^2q^4 + 6pq^5 + q^6$$

Outcome when three coins are tossed $(p+q)^3$: Terms of binomial expression

Let p= heads; q=Tails; p/q= ½



Event	No. of heads	Symbolised by	Probability
HHH	3	p^3	(1/2) 3 = 1/8
HHT	2	$3p^2q$	3(1/2) 2(1/2)=3/8
HTH			
THH			
HTT	1	$3pq^2$	3(1/2) (1/2)2 = 3/8
THT			
TTH			
TTT	0	q^3	(1/2)3= 1/8

$$(p + q)^3 = p^3 + 3p^2q + 3pq^2 + q^3$$

(1/8 +3/8 +3/8 + 1/8) =1

Example 1:

If n=5, p=2/4 calculate mean, variance of binomial distribution.

Solution:

q+p=1

Therefore 2/4+2/4=1

q=2/4

Mean= *np* = 5* 2/4= 2.5

Variance= *nPq*=5*2/4*2/4 **= 1.25**

Example 2:

20% of trees in a forest are infested with a fungus and 4 trees are randomly sampled for infection. Obtain the probability of distribution of X(the no. of trees having infection).

Solution:

Let p represent the probability of X. p=(0.2) (Probability of infected trees); q=(0.8)(probability of normal trees)

X	**Probability with infection f(X)**
0	$^4C_0p^0q^4=(o.2)^0(0.8)^4$
1	$^4C_1p^1q^3 = 4 \times (o.2)(0.8)^3$
2	$^4C_2p^2q^2 = 6 \times (o.2)^2(0.8)^2$
3	$^4C_3p^3q = 4 \times (0.2)^3(0.8)$
4	$^4C_4p^4 = (o.2)^4$
Total	1.000

Example 3:

Of 8 crop plants in a glass room, 2 show bacterial infection and the remaining show viral infection. What is the probability that 5 plants show bacterial infection?

Solution:

^nCxp^xQ^n-x

p=bacterial infection; q=viral infection (where p=2/8, q= 6/8)

using the above formula,

$= {}^8C_5(2/8)^5\,(6/8)^3$ $({}^8C_5 = 8!/\,5! = 336)$

$=336 * (1/4)^5\,(3/4)^3$

Using this information, calculate the probability that no more than 2 in a batch of 10 plants get viral infection.

P(x=0) + P(x=1) + P(x=2), n=10

Using binomial formula,

P(x= no more than 2)

${}^{10}C_0(2/8)^0(6/8)^{10}+\,{}^{10}C_1(2/8)^1(6/8)^9+\,{}^{10}C_2(2/8)^2(6/8)^8$

➢ Poisson Distribution:

Any random variable occurring independently with small probability follows the Poisson distribution.

Some examples are:

1. The no. of RBC distributed in squared of haemocytometer.
2. The distribution of parasites in a given host.
3. Occurrence of accidents on a given day.
4. The rate of mutations.

Poisson distribution is one of the important discrete distribution used widely in biology, medicine and bioinformatics.

The Poisson distribution can be derived as a limiting form of the binomial distributioninder the following conditions.

1. n, the no. of trials of an experiment is very large.
2. p, the probability of occurrences of the event is very small.
3. np= mean of distribution is a finite quantity.

If X is the number of occurrences of some random event in an interval of time or space, the probability tha X will occur is given by

$$\frac{e^{-\lambda}\lambda^{x}}{x!}$$

e = Euler's constant ≈ 2.718
λ = mean or expected value of the variable
x = number of success for the event
! = factorial

Problem:

The mean number of bacteria per milliliter of a liquid is known to be 0.5. Assuming that the number of bacteria follows a Poisson distribution, find the probability that, in 1 ml of liquid there be (a) 1 Bacteria (b) 4 Bacteria (c) 6 Bacteria (d) 8 Bacteria.

$F(x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$

λ = (Mean) = 0.5

$f(x) = 1$

($e^{-½}$ = 0.607) refer table (or) $\frac{1}{\int 2.71828}$ = 0.607

$P(1) = \frac{e^{-\lambda}\lambda^{1}}{1!}$ = ½ $e^{-½}$ = 0.3033

$P(4 = \frac{e^{-\lambda}\lambda^{4}}{4!} = \frac{e^{-½(½)4}}{4x3x2x1}$ = 0.0758

$P(6) = \frac{e^{-\lambda}\lambda^{3}}{6!} = \frac{e^{-½(½)6}}{6x5x4x3x2x1}$ = 0.0126

$P(8) = \frac{e^{-\lambda}\lambda^{8}}{8!} = \frac{e^{½(½)8}}{8x7x6x5x4x3x2x1}$

➢ Normal Distribution

- It is a continuous distribution.
- Major contribution by Carl Gauss, therefore also called as Gaussian distribution to honor him.
- It is the distribution of variables.
- The normal distribution can be derived as a limiting form of the binomial distribution.

The normal density id given by

$$f(X) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(X-\mu)^2/2\sigma^2} \quad -\alpha < x < \alpha$$

π = 3.14

e= 2.718

μ= mean

σ= Standard deviation

Properties of Normal Curve

1. The mean, median and mode of the distribution coincide.

2. The curve is symmetric about the line drawn perpendicular to x-axis at X=μ.

3. The total area under the normal probability curve from –α to + α is 1.

4. As X increases on either side of the mean the frequency curve or the probability falls down without , however, touching the x-axis.

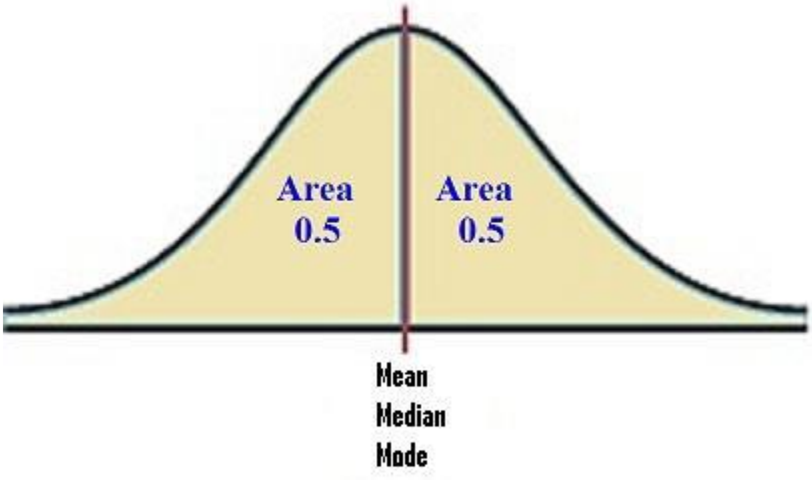


Figure 1: Frequency curve for normal probability distribution

5. Area property

I. P(μ ± 1σ)= 68.27%

II. P(μ ± 2σ)=95%

III. P(μ ± 3σ)=99.7%

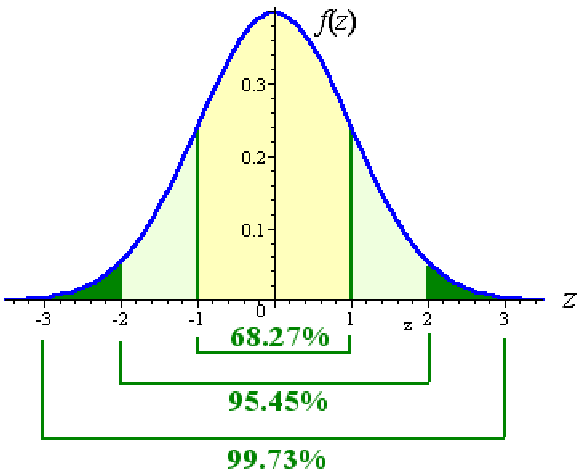


Figure 2: Areas under normal curve

6. The important parameters of normal distribution are μ and σ. As these values change the shape of normal curve changes.